On the Accuracy of Loss Estimates

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In telephone traffic studies, the observed proportion of unsuccessful attempts over a given time interval is one of the measures commonly used to evaluate the grade of service provided by trunk groups. This paper deals with the derivation of an approximate formula for the variance of this estimate when (i) call arrivals constitute a Poisson process, (ii) service times are independent of each other and identically distributed according to a negative exponential law, and (iii) calls placed when all trunks are busy are either cancelled or sent via some alternate route (loss system). Comparison of simulation data with numerical values computed by means of this formula indicates that the latter is accurate enough for practical purposes.

The observed proportion of time during which all trunks are occupied is also an estimate of the grade of service (defined as the probability that a call will be lost or overflow). It is shown here, that for relatively small loads, this estimate has a smaller variance than the observed proportion of lost or rerouted calls. However, as the load is increased, the inequality between the variances of these two estimates is reversed, the cross-over occurring in the vicinity of the point where the load (in erlangs) is equal to the number of trunks.

For a given observation period, the proportion of time when all trunks are busy can be either measured exactly or estimated by "switch-counting." In the latter case, the group is scanned at regular intervals and one observes, for each scan, whether all trunks are occupied or not. The average number of scans which indicate that all trunks are busy is an estimate of this proportion and is, a fortiori, an estimate of the probability of loss. The effect of the scanning rate on the accuracy of this estimate is investigated.

I. INTRODUCTION

In this paper, we shall consider the simplest type of loss systems, namely full availability groups with Poisson inputs, negative exponential service times, and cancellation or rerouting of calls finding all trunks occupied. Under these assumptions, we shall obtain an approximate expression for the variance of the measured call congestion, the latter being defined here as the proportion of calls which either are lost or overflow to some alternate group during a given time interval. In the derivation of this expression, use is made of the classical formula for the propagation of errors, whose computation requires the evaluation of the first- and second-order moments of the joint distribution of the number of offered and the number of overflow calls. Since the marginal means and variances of this distribution are known (cf. Ref. 1), the emphasis is placed here on the derivation of the covariance. Computed values of the variance of the measured call congestion are shown to be in good agreement with simulation results (cf. Figs. 1–5).* Charts giving the variance of this ratio for group sizes up to 50 and offered loads (in erlangs) per trunk of 0.1 to 10, are reproduced in Figs. 6–8.*

For a given observation period, the measured call congestion and the observed proportion of time when all servers are busy — here called measured time congestion — provide us with two estimates of the probability that a call will either be lost or overflow to some alternate route. Neither of these two estimates has a uniformly smaller variance than the other. Actually, the following holds: for relatively small loads, the measured time congestion has a smaller variance than the measured call congestion. However, as the load is increased, the direction of the inequality is reversed, the cross-over occurring in the vicinity of the point where the load (in erlangs) is equal to the number of trunks. Thus, the measured time congestion is not always a more efficient estimator of the probability of loss than the measured call congestion (cf. Figs. 9 and 10).

(In what follows, the terms measured time congestion and measured call congestion will always be abbreviated to time and call congestion, respectively. These terms will refer throughout to measurements performed over a given time interval.)

For a given observation period, time congestion can be either measured exactly or estimated by switchcounting. In the latter case, the group is scanned at regular intervals and one observes, for each scan, whether all trunks are busy or not. The proportion of scans which indicate that all trunks are busy is an unbiased estimate of time congestion and is, a fortiori, an estimate of the probability of loss. Clearly, the variance of this estimate increases as the scanning rate decreases. The loss of accuracy due to scanning is depicted in Fig. 11.

* See illustrations placed later in this article.

Under the present assumptions, loss probabilities can also be estimated from carried loads measured either exactly or by scanning. For offered loads (in erlangs) falling short of the number of trunks, simulation has shown that such estimates have smaller variances than the estimates mentioned earlier. This fact is illustrated in Fig. 12. The effect of scanning on the accuracy of loss estimates based on load measurements is sketched in the same figure.

Finally, we note that estimates of loss probabilities based either on observed call congestion or on carried load measurements are biased, respectively, downwards and upwards. These biases are, however, quite small and likely to be negligible in most situations of practical interest.

II. THE COVARIANCE FUNCTION

Consider a group of c trunks which operate in parallel and are fully available to all requests. If a call is placed when a trunk is free, service starts immediately; otherwise the request is either cancelled or routed via some alternate group (loss system). Regarding the input and the service durations, the following assumptions will be made:

(*i*) The time intervals between successive service demands (whether successful or not) are independent of each other and have a common negative exponential distribution with mean equal to $1/a$ (Poisson input).

(*ii*) The service times are independent of each other and have a common negative exponential distribution whose mean will be taken throughout as the unit of time (a is therefore the offered load in erlangs).

The following notation will be used:

$$\begin{aligned} N(t) &= \text{number of busy trunks at time } t, \\ R(t) &= \text{total number of requests offered during } (0,t), \\ S(t) &= \text{total number of unsuccessful requests during } (0,t), \\ P(t,n,r,s) &= \Pr\,[N(t) = n,\ R(t) = r,\ S(t) = s]. \end{aligned}$$

From the definition of $P(\cdot,\cdot,\cdot,\cdot)$, it follows that:

$$\begin{aligned} P(t,n,r,s) &= 0 \text{ for } n > c, \qquad (t \geqq 0) \\ P(t,n,r,s) &= 0 \text{ for } s > r, \qquad (t \geqq 0) \\ P(0,n,r,s) &= 0 \text{ for } r \geqq 1. \end{aligned}$$

It will be convenient to extend the definition of $P(\cdot,\cdot,\cdot,\cdot)$ and to adopt the convention:

pansion of F_1 in powers of x and the summation on the right of (6) is the coefficient of x^c in that same expansion. Therefore

$$G(w,y,z) = -\frac{(c+1)\tau_{c+1}(w,y) - ayz\cdot\tau_c(w,y)}{(c+1)\sigma_{c+1}(w,y) - ayz\cdot\sigma_c(w,y)} \tag{7}$$

where the τ's and σ's are defined by

$$e^{ayx}(1-x)^{a(y-1)-w} = \sum_0^\infty \sigma_n(w,y)x^n \tag{8}$$

$$Ke^{(x-1)ay}(1-x)^{a(y-1)-w}\int_0^{1-x} u^{a(1-y)+w-1}e^{a(y-1)u}\,du = \sum_0^\infty \tau_n(w,y)\,x^n. \tag{9}$$

We note that if $c_n(\cdot,\alpha)$ and $L_n^{(\alpha)}(\cdot)$ stand respectively for the Poisson-Charlier and the Laguerre polynomials of degree n and parameter α; i.e., if (Ref. 2, pp. 34–35 and 101, and Ref. 3, p. 26)

$$c_n(t,\alpha) = \alpha^{n/2}(n!)^{\frac{1}{2}}\sum_{\nu=0}^{n}(-1)^{n-\nu}\binom{n}{\nu}\alpha^{-\nu}t(t-1)\cdots(t-\nu+1)$$

and

$$L_n^{(\alpha)}(t) = \sum_{\nu=0}^{n}\binom{n+\alpha}{n+\nu}\frac{(-t)^\nu}{\nu!}.$$

then:

$$e^{\alpha x}(1-x)^t = \sum_0^\infty c_n(t,\alpha)\,[(\alpha x)^n/n!]$$

$$= \sum_0^\infty (-1)^n L_n^{(t-n)}(\alpha)x^n$$

and

$$\sigma_n(w,y) = [(ay)^n/n!]\,c_n[a(y-1)-w,ay]$$

$$= (-1)^n L_n^{[a(y-1)-w-n]}(ay).$$

[The relation between the σ's and Kosten's φ-functions (cf. Ref. 1) is readily found. Indeed, by definition

$$e^{a(x-1)}(1-x)^z = \sum_0^\infty \varphi_n^z x^n$$

so that

$$\sigma_n(w,y) \equiv e^{ay}\varphi_n^{a(y-1)-w}.]$$

For later purposes we note that:

$$\sigma_{m+1}(w,y) = \sigma_{m+1}(w+1,y) - \sigma_m(w+1,y), \quad (m = 0, 1, \cdots) \tag{10}$$

$$[w - a(y-1)]\sigma_m(w+1,y) = (m+1)\sigma_{m+1}(w,y) - ay\cdot\sigma_m(w,y), \quad (m = 0, 1, \cdots) \tag{11}$$

$$\sum_0^c \sigma_n(w,y) = \sigma_c(w+1,y). \tag{12}$$

These identities are immediate consequences of recurrence relations known to hold for the corresponding Laguerre polynomials (cf. Ref. 2, p. 98).

Substituting (7), (8) and (9) into (5) yields:

$$F_1(w,x,y,z) = -\frac{(c+1)\tau_{c+1}(w,y) - ayz\cdot\tau_c(w,y)}{(c+1)\sigma_{c+1}(w,y) - ayz\cdot\sigma_c(w,y)} \sum \sigma_n(w,y)x^n + \sum \tau_n(w,y)x^n. \tag{13}$$

We can now obtain the generating function, $F(\cdot,\cdot,\cdot)$, of the Laplace transforms of the joint probabilities Pr $[R(t) = r, S(t) = s]$, $r,s = 0, 1, \cdots$, by deleting from (12) all terms of degree higher than c in x and then setting x equal to 1. If we perform these operations and then make use of (12), we find that:

$$F(w,y,z) = -\frac{(c+1)\tau_{c+1}(w,y) - ayz\cdot\tau_c(w,y)}{(c+1)\sigma_{c+1}(w,y) - ayz\cdot\sigma_c(w,y)}\cdot\sigma_c(w+1,y) + \sum_0^c \tau_n(w,y). \tag{14}$$

The moments of the joint distribution of R and S can now be obtained by evaluating the derivatives of $F(w,\cdot,\cdot)$ for $y = z = 1$.

Differentiating (7) with respect to z and making use of (11), we find

$$\left.\frac{\partial F}{\partial z}\right|_{z=1} = \frac{ay\cdot\tau_c(w,y)}{w - a(y-1)} - \frac{ay\cdot\sigma_c(w,y)[(c+1)\tau_{c+1}(w,y) - ay\cdot\tau_c(w,y)]}{[w - a(y-1)]^2\,\sigma_c(w+1,y)}. \tag{15}$$

In particular, for $y = 1$, we have the well-known result

$$\left.\frac{\partial F}{\partial z}\right|_{y=z=1} = \frac{aE_{1,c}(a)}{w^2}$$

so that

$$ES(t) = at\, E_{1,c}(a) \tag{16}$$

where $E_{1,c}(a)$ is Erlang's loss formula.

Taking the derivative of (15) with respect to y and then setting y equal to 1, yields:

$$\begin{aligned}\left.\frac{\partial^2 F}{\partial y \partial z}\right|_{y=z=1} = {} & \frac{a}{w}\,\tau_c(w)\left[1 + \frac{a}{w} + \frac{a\sigma_c(w)}{w\sigma_c(w+1)}\right] \\ & + \frac{a}{w}\frac{\partial}{\partial y}\,\tau_c(w,y)\mid_{y=1} \\ & - \frac{a\sigma_c(w)}{w^2\sigma_c(w+1)}\left[(c+1)\,\frac{\partial}{\partial y}\,\tau_{c+1}(w,y)\right. \\ & \left. - a\,\frac{\partial}{\partial y}\,\tau_c(w,y)\right]_{y=1}\end{aligned} \tag{17}$$

where

$$\sigma_m(w) \equiv \sigma_m(w,1) \text{ and } \tau_m(w) \equiv \tau_m(w,1), \qquad (m = 0, 1, \cdots).$$

To determine the derivatives of $\tau_c(w,y)$ and $\tau_{c+1}(w,y)$ with respect to y, consider the generating function

$$H(w,x,y) \equiv Ke^{(x-1)ay}\,(1 - x)^{a(y-1)-w} \int_0^{1-x} u^{a(1-y)+w-1}\, e^{a(y-1)u}\, du.$$

Differentiating this expression with respect to y and then setting y equal to 1, we find that

$$\begin{aligned}\frac{\partial}{\partial y}\,H(w,x,y)\mid_{y=1} &= \sum_0^{\infty} \frac{\partial}{\partial y}\,\tau_m(w,y)\,x^m \mid_{y=1} \\ &= K\cdot\frac{ae^{a(x-1)}}{w^2(1+w)}\,(1 + wx)\end{aligned}$$

and, therefore:

$$\frac{\partial}{\partial y}\,\tau_m(w,y)\mid_{y=1} = K\,\frac{ae^{-a}}{(m-1)!w(1+w)}\left(1 + \frac{a}{mw}\right), \qquad (m = 1, \cdots). \tag{18}$$

Since

$$\tau_m(w) = \frac{Ke^{-a}a^m}{w\cdot m!}$$

we obtain, upon taking (18) into account:

$$\mathfrak{L}\{E[R(t)\cdot S(t)]\} = 2K\frac{e^{-a}a^{c+2}}{c!w^3} + K\frac{e^{-a}a^{c+1}}{c!w^2}\cdot\frac{\sigma_{w+2}(c)}{\sigma_{w+1}(c)} \tag{19}$$

where the notation $\mathfrak{L}\{f\}$ is used to designate the Laplace transform of f.

Since $ER(t) = at$ and $ES(t) = at\, E_{1,c}(a)$, we also have:

$$\mathfrak{L}\{\mathrm{Cov}\,[R(t), S(t)]\} = K\frac{e^{-a}a^{c+1}}{c!w^2}\cdot\frac{\sigma_c(w+2)}{\sigma_c(w+1)} \tag{20}$$

where Cov $[R(t),S(t)]$ stands for the covariance between $R(t)$ and $S(t)$.

For $y = 1$, $m = c$ and w replaced by $w + 1$, (11) reduces to

$$(w+1)\sigma_c(w+2) = (c+1)\sigma_{c+1}(w+1) - a\sigma_c(w+1),$$

and (20) can be rewritten as follows:

$$\mathfrak{L}\{\mathrm{Cov}\,[R(t), S(t)]\} = K\frac{e^{-a}a^{c+1}}{c!w^2(w+1)\sigma_c(w+1)}\,[(c+1)\sigma_{c+1}(w+1) - a\sigma_c(w+1)]. \tag{21}$$

Let w_i, $i = 1, \cdots, c$ be the c roots of $\sigma_c(w+1)$. It is well known that these roots are simple, smaller than -1 and at least one unit apart. Then expanding (21) in partial fractions and making use of the relation $(c+1)\sigma_{c+1}(0) - a\sigma_c(0) = 0$, which is (11) for $y = 1$, $w = 0$, we find:

$$\begin{aligned}\mathfrak{L}\{\mathrm{Cov}\,[R(t), S(t)]\} &= K\frac{e^{-a}a^{c+1}}{c!}\Bigg[\frac{(c+1)\sigma_{c+1}(1) - a\sigma_c(1)}{w^2\,\sigma_c(1)} \\ &\quad + \frac{c+1}{w}\frac{\partial}{\partial w}\frac{\sigma_{c+1}\,(w+1)}{\sigma_c(w+1)}\bigg|_{w=0} - \frac{(c+1)\sigma_{c+1}(1) - a\sigma_c(1)}{w\sigma_c(1)} \\ &\quad + (c+1)!\sum_{i=1}^{c}\frac{\sigma_{c+1}(w_i+1)}{w_i^2(1+w_i)(w-w_i)\prod_{j\neq i}(w_i - w_j)}\Bigg]\end{aligned}$$

and the covariance between R and S is, therefore, given by

$$\begin{aligned}\mathrm{Cov}\,[R(t), S(t)] &= K\frac{e^{-a}a^{c+1}}{c!}\Bigg[\frac{(c+1)\sigma_{c+1}(1) - a\sigma_c(1)}{\sigma_c(1)}\cdot t \\ &\quad + (c+1)\frac{\partial}{\partial w}\frac{\sigma_{c+1}(w+1)}{\sigma_c(w+1)}\bigg|_{w=0} - \frac{(c+1)\sigma_{c+1}(1) - a\sigma_c(1)}{\sigma_c(1)} \\ &\quad + (c+1)!\sum_{i=1}^{c}\frac{\sigma_{c+1}(w_i+1)e^{w_i t}}{w_i^2(1+w_i)\prod_{j\neq i}(w_i - w_j)}\Bigg].\end{aligned} \tag{22}$$

To determine explicitly the derivative appearing in (22), let us consider Cov $[R(t),S(t)]$ for small values of t. Writing $P(t,r,s)$ for the (equilibrium) probability that, during a time interval of length t, r requests arrived and that, among these r requests, s of them found all the trunks busy, we have (t small):

$$\begin{aligned} P(t,0,0) &= 1 - at + o(t) \\ P(t,1,0) &= at\,[1 - E_{1,c}(a)] + o(t) \\ P(t,1,1) &= at\,E_{1,c}(a) + o(t) \\ P(t,0,s) &= 0, \qquad (s \geqq 1) \\ P(t,r,0) &= o(t), \qquad (r > 1) \end{aligned}$$

and

$$0 \leqq \sum_{r,s=2}^{\infty} sP(t,r,s) < \sum_{r,s=2}^{\infty} rP(t,r,s) < \sum_{r,s=2}^{\infty} rsP(t,r,s)$$
$$< \sum_{r=2}^{\infty} r^2 e^{-at} \frac{(at)^r}{r!} = o(t).$$

Hence Cov $[R(t),S(t)] = at\,E_{1,c}(a) + o(t)$. Letting t tend to 0 in (22), we find that

$$(c+1) \frac{\partial}{\partial w} \frac{\sigma_{c+1}(w+1)}{\sigma_c(w+1)}\bigg|_{w=0} = \frac{(c+1)\sigma_{c+1}(1) - a\sigma_c(1)}{\sigma_c(1)} - (c+1)! \sum_1^c \frac{\sigma_{c+1}(w_i+1)}{w_i^2(1+w_i) \prod_{j\neq i} (w_i - w_j)}.$$

Substituting this expression in (22) yields

$$\begin{aligned} \text{Cov}\,[R(t), S(t)] = K \frac{e^{-a} a^{c+1}}{c!} \Bigg[& \frac{(c+1)\sigma_{c+1}(1) - a\sigma_c(1)}{\sigma_c(1)}\, t \\ & - (c+1)! \sum_1^c \frac{\sigma_{c+1}(w_i+1)}{w_i^2(1+w_i) \prod_{j\neq i}(w_i - w_j)} \\ & + (c+1)! \sum_1^c \frac{\sigma_{c+1}(w_i+1) e^{w_i t}}{w_i^2(1+w_i) \prod_{j\neq i}(w_i - w_j)} \Bigg]. \end{aligned} \qquad (22')$$

We shall now determine the constant term appearing in (22′). To this end, we note that

$$\sigma_c(w + 1) = \frac{1}{c!}\prod_{i=1}^{c}(w - w_i)$$

and, therefore

$$\frac{\sigma_{c+1}(w+1)}{w(1+w)\sigma_c(w+1)} = \frac{1}{w}\frac{\sigma_{c+1}(1)}{\sigma_c(1)} - \frac{1}{1+w}\frac{\sigma_{c+1}(0)}{\sigma_c(0)} + c!\sum_{i=1}^{c}\frac{1}{(w - w_i)}\frac{\sigma_{c+1}(w_i+1)}{w_i(1+w_i)\prod_{j\neq i}(w_i - w_j)}$$

Hence, for $w = 0$, we have:

$$\begin{aligned}(c+1)!\sum_{i=1}^{c}\frac{\sigma_{c+1}(w_i+1)}{w_i^2(1+w_i)\prod_{j\neq i}(w_i - w_j)} &= (c+1)\lim_{w\to 0}\left[\frac{1}{w}\frac{\sigma_{c+1}(1)}{\sigma_c(1)} - \frac{1}{1+w}\frac{\sigma_{c+1}(0)}{\sigma_c(0)} - \frac{\sigma_{c+1}(w+1)}{w(1+w)\sigma_c(w+1)}\right] \\ &= -a.\end{aligned} \tag{23}$$

Furthermore, for $w = 1 + w_i$, $y = 1$ and $m = c$, (11) yields:

$$(w_i + 1)\sigma_c(w_i + 2) = (c + 1)\sigma_{c+1}(w_i + 1). \tag{24}$$

We also note that the c roots of $\sigma_c(w + 2)$ are $w_i - 1$, $i = 1, \cdots, c$, so that

$$\sigma_c(w + 2) = \frac{1}{c!}\prod_{1}^{c}(w - w_j + 1). \tag{25}$$

Hence, combining (24) and (25), we have:

$$\frac{(c+1)!\sigma_{c+1}(w_i+1)}{1+w_i} = \prod_{j=1}^{c}(w_i - w_j + 1). \tag{26}$$

Using (23), (26) and the relations

$$Ke^{-a}a^{c+1} = c!\,aE_{1,c}(a)$$

$$(c + 1)\sigma_{c+1}(1) = \sigma_c(1)\{c + 1 + aE_{1,c}(a)\}$$

(22′) can be simplified as follows:

$$\operatorname{Cov}[R(t), S(t)] = aE_{1,c}(a)\Bigg[\{c + 1 - a[1 - E_{1,c}(a)]\}t + a - \sum_{i=1}^{c} \frac{e^{w_i t}}{w_i^2} \prod_{j \neq i} \left(1 + \frac{1}{w_i - w_j}\right)\Bigg]. \tag{27}$$

Since, as pointed out above:

$$\max_{1 \leqq i \leqq c} w_i < -1$$

$$\min_{\substack{1 \leqq i,j \leqq c \\ i \neq j}} | w_i - w_j | > 1$$

we have:

$$\sum_{i=1}^{c} \frac{e^{w_i t}}{w_i^2} \prod_{j \neq i} \left(1 + \frac{1}{w_i - w_j}\right) > 0$$

and

$$\frac{\partial}{\partial t} \sum_{i=1}^{c} \frac{e^{w_i t}}{w_i^2} \prod_{j \neq i} \left(1 + \frac{1}{w_i - w_j}\right) < 0.$$

Hence we have the following inequalities [use is also made here of (23)]:

$$aE_{1,c}(a)\{c + 1 - a[1 - E_{1,c}(a)]\}t$$
$$< \operatorname{Cov}[R(t),S(t)]$$
$$< aE_{1,c}(a)\ [\{c + 1 - a[1 - E_{1,c}(a)]\}t + a]$$

and, for large values of $t(>0)$:

$$\operatorname{Cov}[R(t),S(t)] = aE_{1,c}(a)\ [\{c + 1 - a[1 - E_{1,c}(a)]\}t + a] + o(e^{-t}). \tag{28}$$

III. VARIANCE OF CALL CONGESTION

In the preceding section, exact and asymptotic formulas were obtained for the covariance between the number of offered calls, $R(t)$, and the number of overflow calls, $S(t)$, during a time interval of length t. These expressions can now be combined with known formulas for the means and variances of $R(t)$ and $S(t)$ to obtain an approximate expression for the variance of call congestion. Indeed, according to the classical formula for the propagation of errors, we have:

$$\begin{aligned}\text{Var}\,[S(t)/R(t)] &\\ &\sim \{1/ER(t)\}^2\,\text{Var}\,[S(t)] + \{ES(t)/[ER(t)]^2\}^2\,\text{Var}\,[R(t)]\\ &\qquad - 2\{ES(t)/[ER(t)]^3\}\,\text{Cov}\,[R(t),S(t)]\\ &= (1/at)^2\,\text{Var}\,[S(t)] + E_{1,c}^2(a)/(at)\\ &\qquad - 2[E_{1,c}(a)/(at)^2]\,\text{Cov}\,[R(t),S(t)]\\ &< (1/at)^2\,\text{Var}\,[S(t)]\end{aligned} \tag{29}$$

where, assuming t large:

$$\text{Var}\,[S(t)] \sim at\,E_{1,c}(a)\left[1 + 2a\frac{\partial}{\partial w}\frac{\sigma_c(w)}{\sigma_c(w+1)}\right]_{w=0} \tag{30}$$

and

$$\text{Cov}\,[R(t),S(t)] \sim at\,E_{1,c}(a)[c + 1 - a + aE_{1,c}(a)]. \tag{31}$$

We note that Var $[S(t)/R(t)]$ is, asymptotically, of the form k/t, where k depends only on a and c.

The exact and asymptotic expressions for Var $[S(t)]$ were first derived by Kosten, Manning and Garwood.[1] These formulas can be obtained in a straightforward manner from the generating function (14) with y set equal to 1. The asymptotic expression (30), however, is rather involved and its use can be avoided as follows. Indeed, we note that, under the present assumptions, the instants at which overflows occur constitute a renewal process (i.e., the intervals between any pair of consecutive overflows are independent of each other and have the same distribution). Then using Smith's extension of a result due to Feller (cf. Ref. 4, pp. 296–298 and Ref. 5, pp. 30–33), we have:

$$\text{Var}\,[S(t)] \sim [\mu_2(c) - \mu_1^2(c)]t/\mu_1^3(c) \tag{32}$$

where $\mu_n(c)$ is the nth moment of the interoverflow distribution of a group of c trunks.

The expression on the right-hand side of (32) is rather easy to compute, since we have the following recurrence relations (cf. Ref. 6, p. 388)

$$a\mu_2(n) = 2a\mu_1^2(n) + n\mu_2(n-1), \qquad (n = 1,2,\cdots) \tag{33}$$

where $\mu_1^{-1}(c) = aE_{1,c}(a)$. Hence

$$\text{Var}\,[S(t)] \sim \frac{[(c/a)\mu_2(c-1) + \mu_1^2(c)]}{\mu_1^3(c)}\,t.$$

The second moment $\mu_2(c-1)$ can be computed either by repeated use of (33) or by means of the explicit formula

$$\mu_2(c-1) = 2\sum_{n=0}^{c-1} \frac{(c-1)_n}{a^n} \mu_1^2(c-1-n)$$

with

$$(c-1)_0 = 1, \qquad (c-1)_n = (c-1)(c-2)\cdots(c-n), \qquad (n \geqq 1).$$

We note that the renewal theorem used above can be applied as long as the input to the system is recurrent, and the other assumptions made here remain the same. In these more general cases, (32) still holds, but the moments $\mu_1(c)$ and $\mu_2(c)$ satisfy less simple recurrence relations. Indeed, we have then:

$$\mu_1(c)\gamma_{c-1}(1) - \mu_1(c-1) = 0$$

and

$$\mu_2(c)\gamma_{c-1}(1) - 2\mu_1(c)[\mu_1(c-1) - \gamma_{c-1}'(1)] - \mu_2(c-1) = 0$$

where $\gamma_n(\cdot)$ is the Laplace-Stieltjes transform of the interoverflow distribution of a group of n trunks and $\gamma_m'(1)$ stands for the derivative of $\gamma_m(\cdot)$ at 1.

The preceding relations follow immediately from Palm's recurrences (cf. Ref. 3, pp. 36–38, and Ref. 7, pp. 16–22):

$$\gamma_n(s)[1 - \gamma_{n-1}(s) + \gamma_{n-1}(s+1)] = \gamma_{n-1}(s+1), \qquad (n = 1,2,\cdots).$$

The standard deviation of the call congestion computed by means of (29) and (31) to (33) is compared in Figs. 1–5 with simulation results. As may be seen from these graphs, there is good agreement between the theoretical and observed values.

On each one of these charts, two additional curves are also plotted, namely:

(*i*) $\{\mathrm{Var}\,[S(t)]\}^{\frac{1}{2}}/ER(t)$ as a function of the offered load.

This expression is an upper bound for the standard deviation of the call congestion. However, unless the offered load is relatively small, it considerably overestimates this standard deviation.

(*ii*) $\{E_{1,c}(a)[1 - E_{1,c}(a)]\}^{\frac{1}{2}}/\{ER(t)\}^{\frac{1}{2}}$ as a function of the offered load.

This quantity, referred to as the binomial approximation, is a lower bound for the standard deviation of the call congestion. This bound underestimates the latter to such an extent, however, that it is of little if any value.

In view of the agreement between the observed and theoretical variances of the call congestion, the latter are graphed in Figs. 6–8 for $c = 1(1)10(2)\ 20(5)50$ and $0.1 \leqq a/c \leqq 10$. These values pertain to the case $t = 20$. The asymptotic variance of the call congestion for any (sufficiently large) value of t may be obtained by multiplying the variances of Figs. 6–8 by $20/t$.

Simulation results have shown that (29) — with Var $[S(t)]$ and Cov $[R(t),S(t)]$ replaced by their respective asymptotic expressions — give sufficiently accurate values of the variance of the call congestion whenever the length of the observation period, t, is such that the expected number of offered calls, $ER(t)$, is about 40 or more. When $ER(t)$

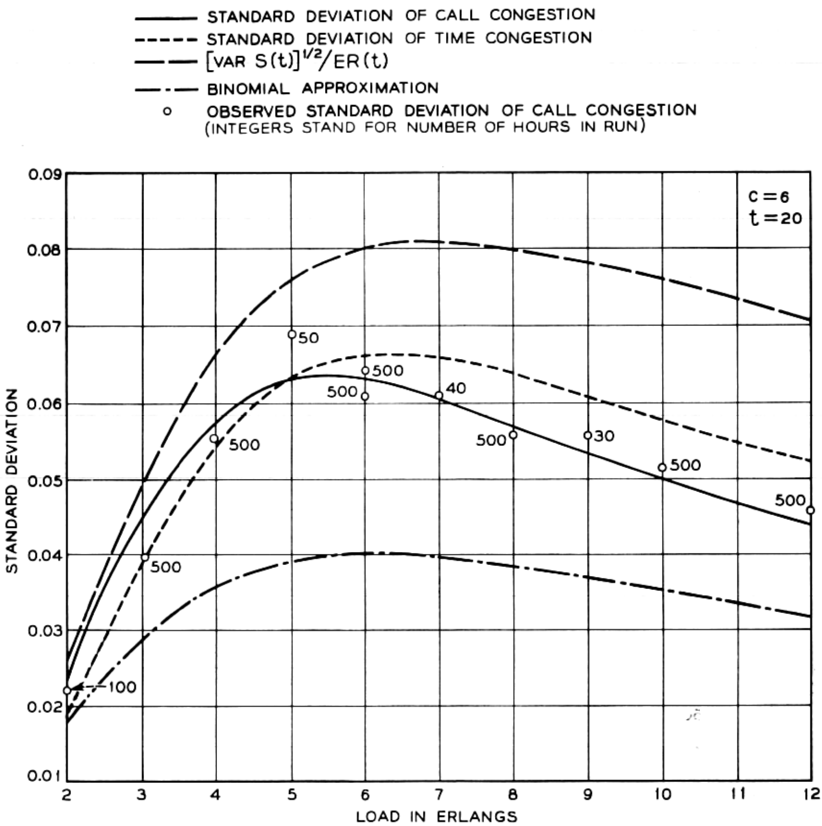

Fig. 1 — Standard deviations of call and time congestions, $c = 6$, $t = 20$.

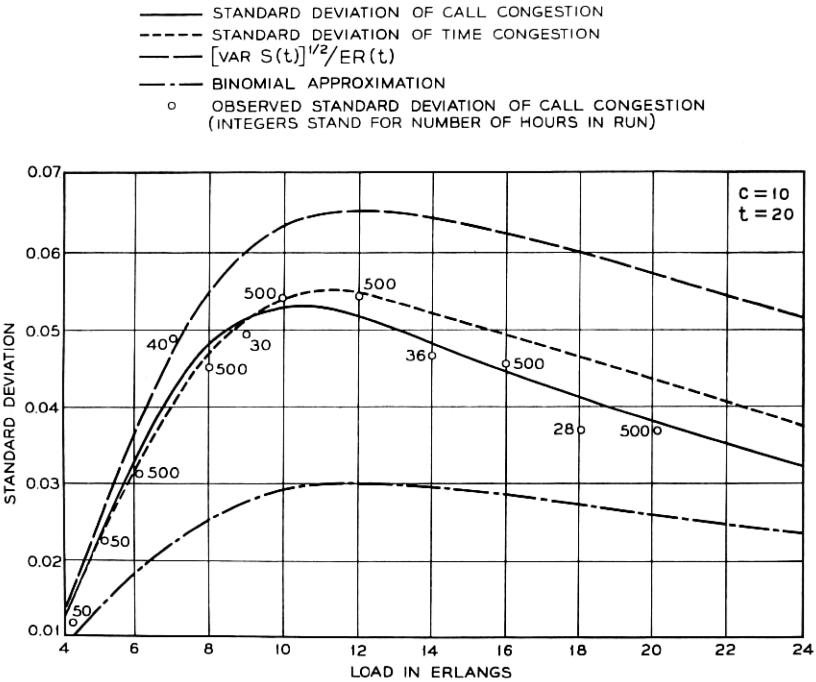

Fig. 2 — Standard deviations of call and time congestions, $c = 10$, $t = 20$.

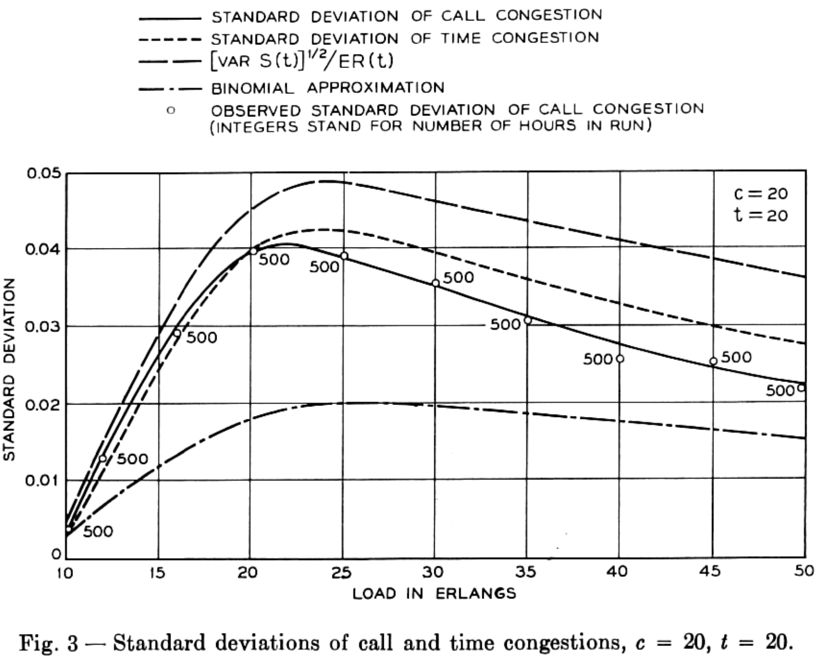

Fig. 3 — Standard deviations of call and time congestions, $c = 20$, $t = 20$.

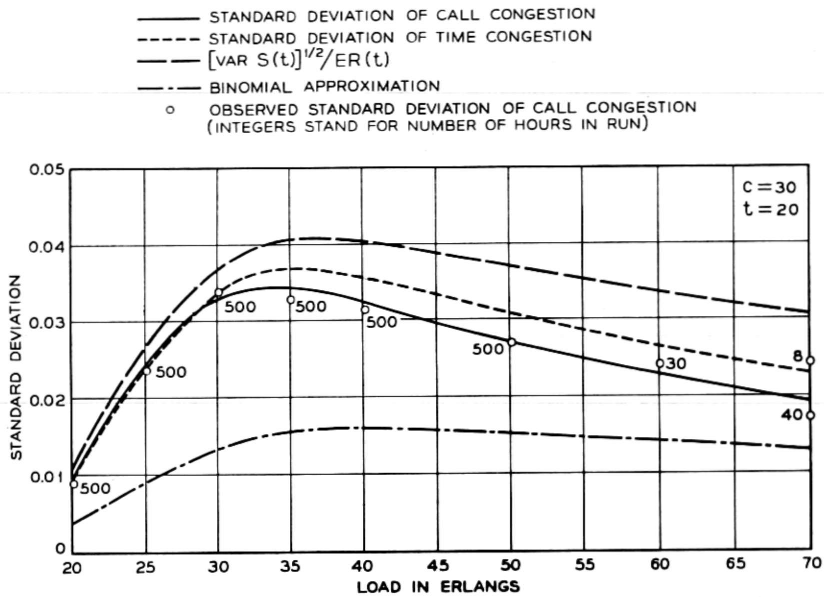

Fig. 4 — Standard deviations of call and time congestions, $c = 30$, $t = 20$.

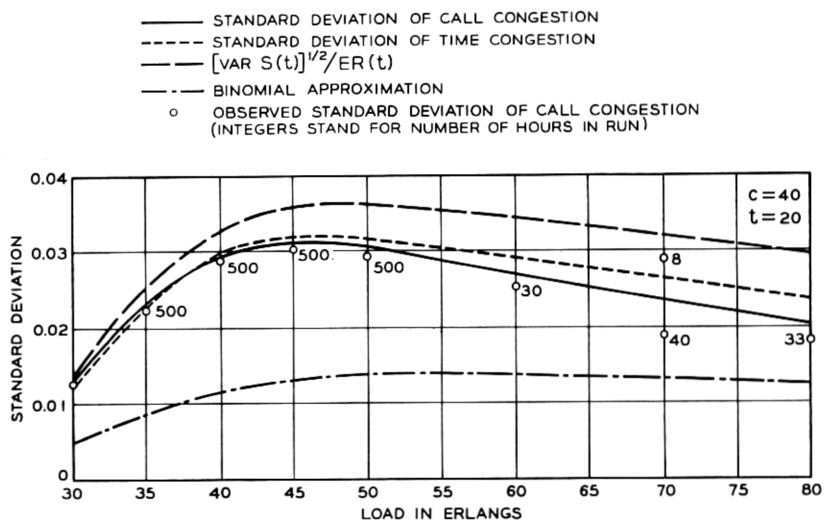

Fig. 5 — Standard deviations of call and time congestions, $c = 40$, $t = 20$.

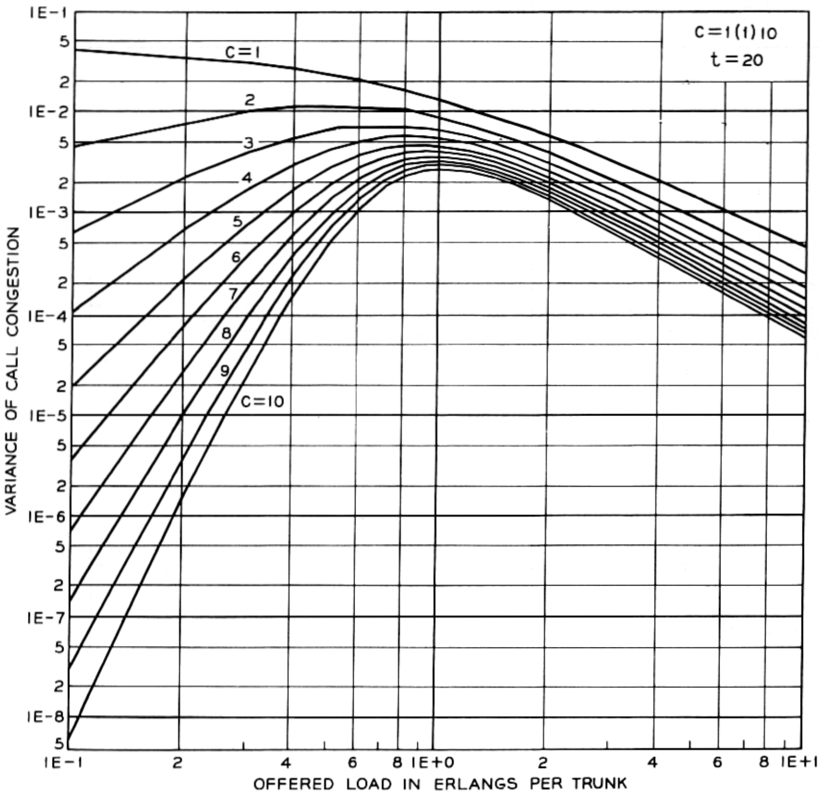

Fig. 6 — Variance of call congestion. Observation period = 20 holding times $c = 1(1)10$.

falls below 40, (29) provides us with an upper bound which becomes increasingly coarse as the expected number of offered calls decreases.

IV. RELATIVE ACCURACY OF LOSS ESTIMATES

Various measurements can be used to estimate the probability of loss. The principal ones are:

(*i*) *The number of offered calls and the number of lost* (*or overflow*) *calls*. The ratio of the latter to the former (i.e., the call congestion) is an asymptotically unbiased estimate of the probability of loss.

(*ii*) *The time congestion*. This quantity, which is an unbiased estimate of the probability of loss, may be either measured exactly or estimated by scanning the trunks at regular intervals and observing, at each scan, how many trunks are busy. The proportion of scans which indicate that all trunks are busy is also an unbiased estimate of the probability of loss.

(*iii*) *The carried load (i.e., the average number of busy trunks) obtained either by continuous observation or by scanning at regular intervals.* This last measurement consists in observing, at regular intervals, the number of busy trunks. The average of these numbers, for a given number of scans, is an unbiased estimate of the carried load. If $\hat{L}$ stands for the carried load measured either exactly or by scanning, then the demand rate, $\hat{a}$, may be estimated by means of the formula

$$\hat{L} = \hat{a}[1 - E_{1,c}(\hat{a})].$$

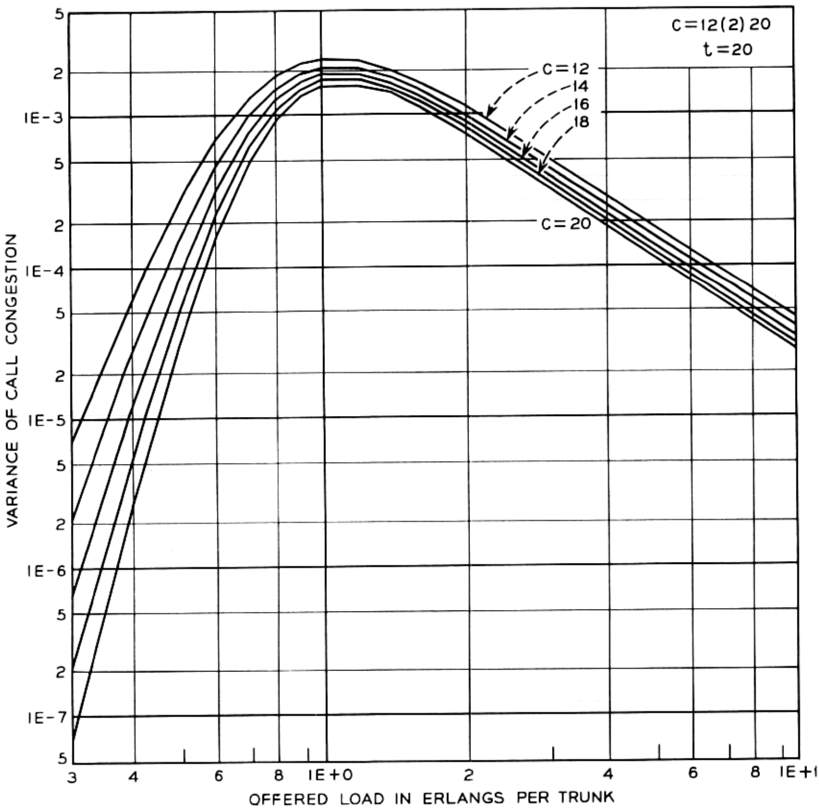

Fig. 7 — Variance of call congestion. Observation period = 20 holding times $c = 12(2)20$.

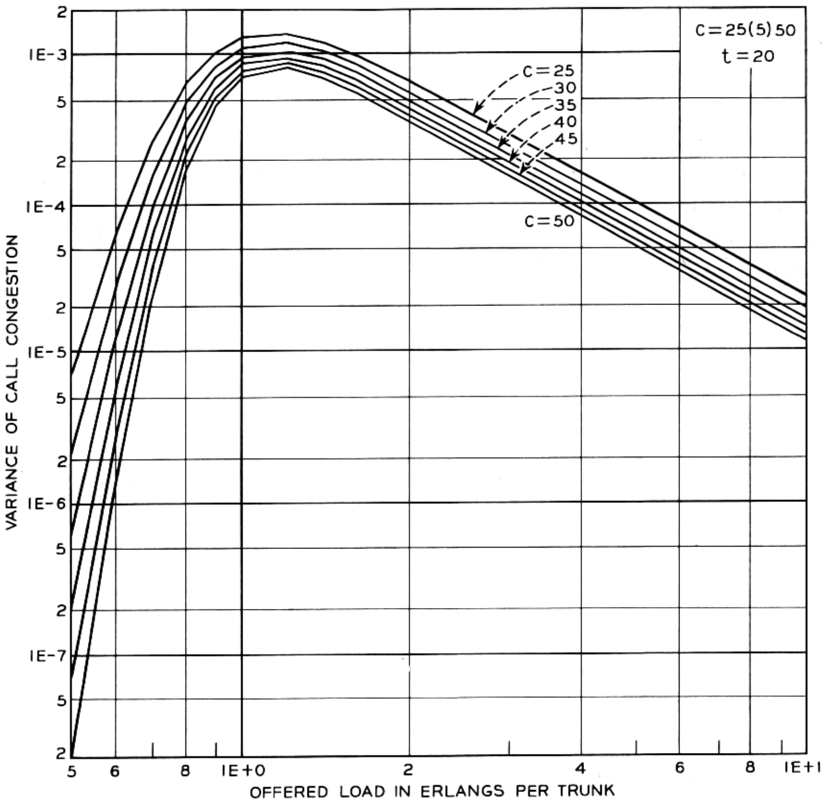

Fig. 8 — Variance of call congestion. Observation period = 20 holding times $c = 25(5)50$.

$E_{1,c}(\hat{a})$ itself is an estimate of the probability of loss. This estimate has a small positive bias which tends to zero as the length of the observation period gets large.

Theoretical as well as observed (simulation) values of the standard deviations of these estimates are plotted in Figs. 9–12. (For each load, the simulation results given in Fig. 12 were computed from a single run of 500 hours. The numerator and the denominator of each ratio appearing in Figs. 9 and 10 were evaluated from a single 500-hour run of simulated traffic.) These graphs reveal typical patterns, namely:

(*i*) When the offered load, in erlangs, falls short of the number of trunks, the loss estimates based on continuous load measurements have smaller standard deviations than both the call and the time

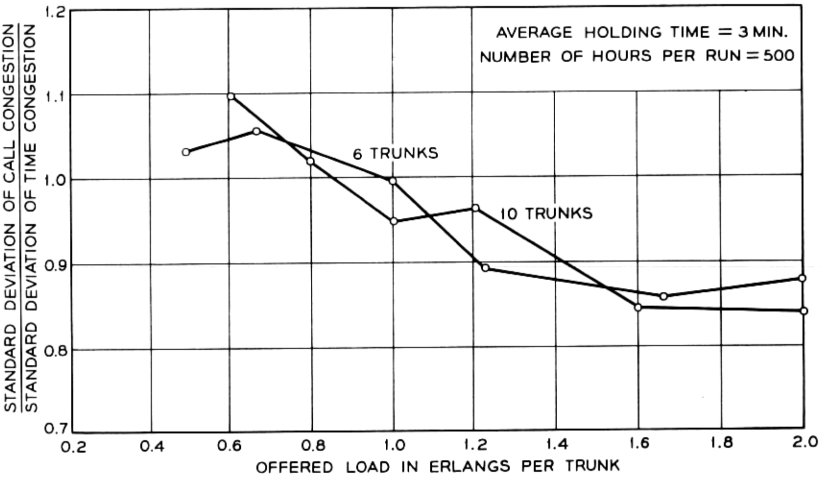

Fig. 9 — Relative accuracy of grade of service estimates based on hourly measurements of call and time congestions — simulation results.

congestions. In the same range, the call congestion has a larger standard deviation than the time congestion.

(*ii*) When the offered load exceeds the number of trunks, the converse situation holds; i.e., the call congestion has a smaller standard deviation than the time congestion, and the standard deviation of the

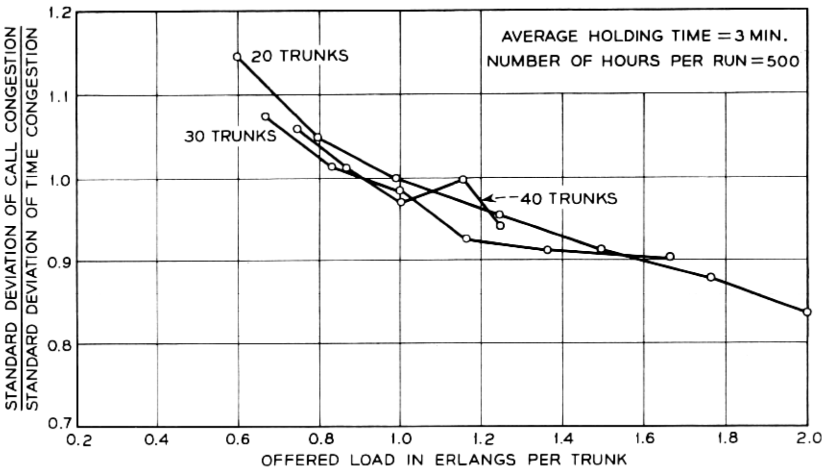

Fig. 10 — Relative accuracy of grade of service estimates based on hourly measurements of call and time congestions — simulation results (cont.).

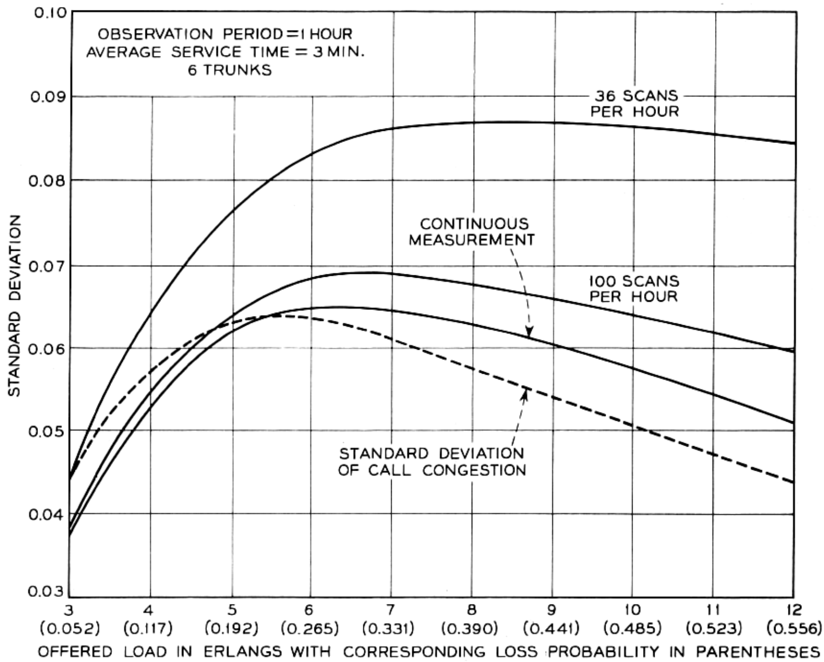

Fig. 11 — Standard deviation of the time congestion estimated by switch-counting.

latter, in turn, is exceeded by the standard deviation of loss estimates based on continuous carried load measurements.

The effect of scanning on the variances of the time congestion and of the loss estimates based on carried load measurements is illustrated in Figs. 11 and 12.

Let us assume now that the length of the observation period is such that (29) closely approximates Var $[S(t)/R(t)]$. Under these conditions, the load beyond which the time congestion is less accurate (in terms of its variance) than the call congestion is approximately equal to the load a determined by the following equation:

$$1 + E_{1,c}(a) = 2E_{1,c}(a)[c + 1 - a + aE_{1,c}(a)]. \qquad (34)$$

This condition is readily seen to be equivalent to the requirement

$$\text{Var } [S(t)/R(t)] = \text{Var } B(t) \qquad (35)$$

where $B(t)$ stands for the time congestion in an observation period of length t. Equations (35), (29), and (31) together with the relations (cf. Ref. 3 p. 131)

$$ES(t) = atEB(t)$$

$$\text{Var } S(t) - ES(t) = (at)^2 \text{ Var } B(t)$$

imply (34).

For given c, (34) has a unique positive root, r, which is smaller than c except in the case $c = 1$ where the root is equal to 1. Computations show that this root lies relatively close to c (cf. Fig. 13).

Let $B_n(t)$ be the estimate of the time congestion obtained by switch-

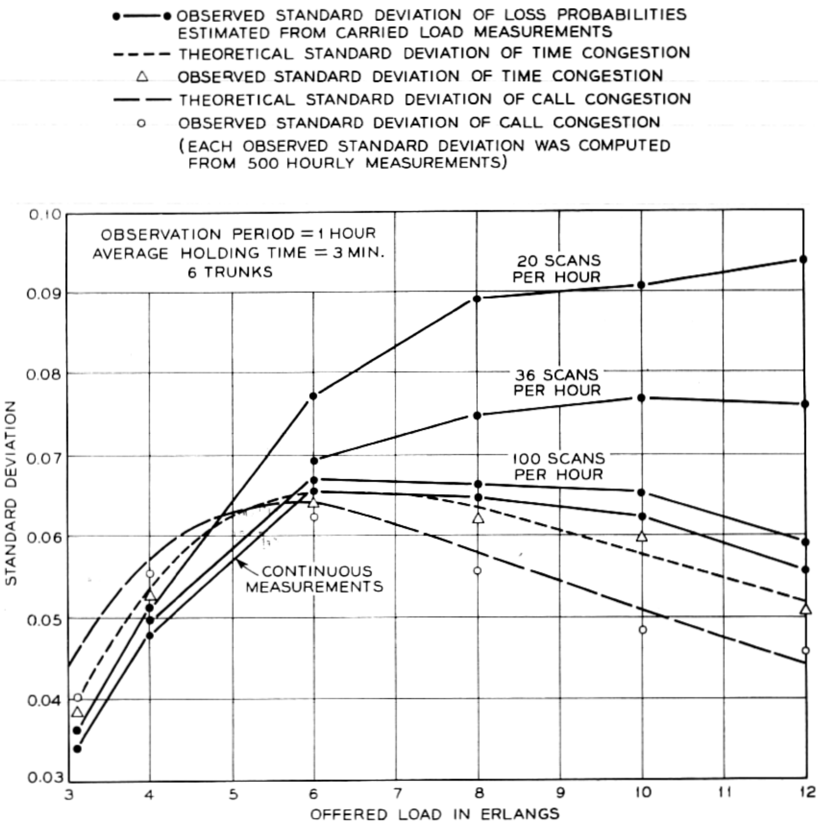

Fig. 12 — Relative accuracy of loss estimates based on call and time congestions and on carried load measurements.

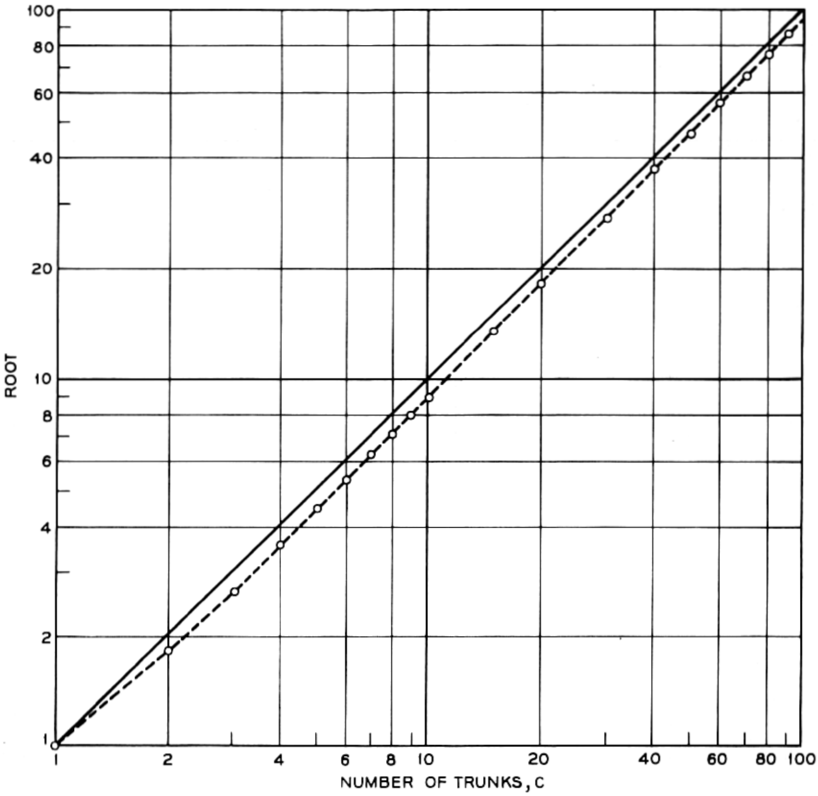

Fig. 13 — Root of equation (34).

counting at the rate of n scans per observation period of length t. We shall now derive an explicit formula for the variance of $B_n(t)$.

Let τ be the interval separating consecutive scans, $N(u)$ be the number of busy trunks at time u, and

$$X(u) = \begin{cases} 1 & \text{if} \quad N(u) = c \\ 0 & \text{if} \quad N(u) < c. \end{cases}$$

Then

$$B_n(t) = n^{-1} \sum_1^n X(i\tau)$$

and

$$EB_n(t) = E_{1,c}(a)$$

$$\text{Var } B_n(t) = n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{n} \text{Cov } [X(i\tau), X(j\tau)].$$

Now let

$$P(u) = Pr\,[N(u) = c \mid N(0) = c].$$

The function $P(\cdot)$, which is called the recovery function of the process $N(\cdot)$, has the following expression (cf. Ref. 3, p. 85 and Ref. 8, p. 135):

$$P(u) = E_{1,c}(a) - \sum_{j=1}^{c} \frac{e^{w_j|u|}}{w_j} \prod_{i \neq j} \left(1 - \frac{1}{w_j - w_i}\right)$$

where, as before, w_i, $i = 1, \cdots, c$, are the c roots of $\sigma_c(w + 1)$. Since

$$\text{Cov } [X(u + v), X(v)] = E_{1,c}(a)P(u) - E_{1,c}^{\,2}(a)$$

we have (cf. Ref. 8, pp. 136–138)

$$\begin{aligned} \text{Var } B_n(t) &= n^{-2}\, E_{1,c}(a) \sum_{i=1}^{n} \sum_{j=1}^{n} P(|\, i - j \mid \tau) - E_{1,c}^{\,2}(a) \\ &= -\, n^{-2}\, E_{1,c}(a) \sum_{-n}^{n} (n - |\, k \,|) \sum_{j=1}^{c} \frac{e^{|k|w_j\tau}}{w_j} \\ &\quad \cdot \prod_{i \neq j} \left(1 - \frac{1}{w_j - w_i}\right) \qquad (36) \\ &= n^{-1}\, E_{1,c}(a) \sum_{j=1}^{c} \left\{\frac{1}{w_j} \prod_{i \neq j} \left(1 - \frac{1}{w_j - w_i}\right)\right\} \\ &\quad \cdot \left\{\text{ctnh}\left(\frac{\tau w_j}{2}\right) + \left(\frac{1 - e^{n\tau w_j}}{2n}\right) \text{csch}^2\left(\frac{\tau w_j}{2}\right)\right\}. \end{aligned}$$

If we let n tend to infinity in this formula, we obtain, in the limit, the variance of the time congestion, $B(t)$, for continuous observation (measurement):

$$\begin{aligned} \text{Var } B(t) = \frac{2E_{1,c}(a)}{t} &\left\{\sum_{j=1}^{c} w_j^{-2} \prod_{i \neq j} \left(1 - \frac{1}{w_j - w_i}\right)\right. \\ &\left. + \frac{1}{t} \sum_{j=1}^{c} w_j^{-3}\, (1 - e^{tw_j}) \prod_{i \neq j} \left(1 - \frac{1}{w_j - w_i}\right)\right\}. \end{aligned}$$

This last formula was first obtained, in a slightly different form, by Kosten, Manning and Garwood (cf. Ref. 1).

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