Dynamics Analysis of a Two-Body Gravitationally Oriented Satellite

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The rigid body motion of a two-body satellite under the action of gravitational torques is analyzed. The satellite consists of two rigid bodies connected by a universal joint where damping is provided in the two journals. The motion of the satellite relative to the mass center thus has five degrees of freedom, two of which are provided with energy dissipation. It appears that the rigid body motion of such a composite satellite will automatically converge upon a motion in which a given axis of the satellite is earth-pointing.

The equations of motion are derived directly from those of Euler. Necessary stability criteria are established. Numerical solutions for a practical scheme are presented.

I. INTRODUCTION

This paper deals with the analysis of the rotational motion of a satellite consisting of two rigid bodies connected by a hinge mechanism of universal joint type. The rotational motion of the satellite thus has five degrees of freedom; the two degrees of freedom that involve the relative motion between the two bodies are provided with energy dissipation. It is found that any motion of the satellite with respect to the local vertical always involves relative motion between the two bodies. Therefore, the damping at the hinge joint dissipates not only the relative motion of the two bodies but also the motion of the satellite with respect to the local vertical. The satellite will then converge upon a stable motion in which a specified axis of the satellite will remain close to the local vertical.

The equations of motion are derived directly from those of Newton and Euler. This approach naturally suggests several additional dependent variables and results in numerically workable equations. This is not the case in the Lagrangian formulation.

There are several practical problems involved in this scheme of pas-

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sive gravitational orientation. One problem is to make the gravitational torque dominate over all other disturbing torques. A novel solution to this problem, which employs extensible rods, has been given by Kamm.[1] Another problem is the development of the hinge dissipative mechanism. A viscous mechanism is described by Kamm,[1] whereas a hysteresis mechanism is suggested in this paper. These practical matters are not the substance of this paper; they are used as illustrations for the numerical treatment of a practical design.

II. GENERAL EQUATIONS OF MOTION

Consider a satellite which is constructed of two rigid bodies, with masses m_1 and m_2, hinged at a point H. The centers of mass of the two bodies are denoted S_1 and S_2, and the center of mass of the composite satellite is denoted S_0. Let the earth's center be O and let P_1 and P_2 be arbitrary points of body 1 and 2. Also, denote $OP_1 = \mathbf{R}_1$, $OS_1 = \boldsymbol{\varrho}_1$, $OS_0 = \boldsymbol{\varrho}$, $OS_2 = \boldsymbol{\varrho}_2$, $OP_2 = \mathbf{R}_2$, $S_1P_1 = \mathbf{r}_1$, $S_2P_2 = \mathbf{r}_2$, $HS_1 = \mathfrak{L}_1$, $HS_2 = \mathfrak{L}_2$ (see Fig. 1). (Note: $\mathfrak{L}_1$ and $\mathfrak{L}_2$ represent vectors, while ℓ_1 and ℓ_2 which appear later represent their respective magnitudes. See Appendix for list of symbols.)

Let us introduce the following notations:

$\boldsymbol{\omega}_{\mathrm{I}}$, $\boldsymbol{\omega}_{\mathrm{II}}$ = angular velocity of body 1, 2,

$\mathbf{T}_H$ = reactive torque transmitted through the joint on body 1,

$\mathbf{F}_H$ = reactive force transmitted through the joint on body 1,

$\mathbf{T}_1$, $\mathbf{T}_2$ = resultant torque on body 1, 2 exclusive of $\mathbf{T}_H$,

$\mathbf{F}_1$, $\mathbf{F}_2$ = resultant force on body 1, 2 exclusive of $\mathbf{F}_H$,

m_1, m_2 = mass of body 1, 2,

$\bar{m} = m_1m_2/(m_1 + m_2)$ = reduced mass of the system,

$m = m_1 + m_2$ = total mass of the system,

$\boldsymbol{\Phi}_1$, $\boldsymbol{\Phi}_2$ = moment of inertia dyadic of body 1, 2.

Newton's and Euler's equations can now be written as

$$\mathbf{F}_1 + \mathbf{F}_H = m_1\ddot{\boldsymbol{\varrho}}_1 \tag{1a}$$

$$\mathbf{F}_2 - \mathbf{F}_H = m_2\ddot{\boldsymbol{\varrho}}_2 \tag{1b}$$

$$\boldsymbol{\Phi}_1\cdot\dot{\boldsymbol{\omega}}_{\mathrm{I}} + \boldsymbol{\omega}_{\mathrm{I}} \times \boldsymbol{\Phi}_1\cdot\boldsymbol{\omega}_{\mathrm{I}} = \mathbf{T}_1 + \mathbf{T}_H - \mathfrak{L}_1 \times \mathbf{F}_H \tag{1c}$$

$$\boldsymbol{\Phi}_2\cdot\dot{\boldsymbol{\omega}}_{\mathrm{II}} + \boldsymbol{\omega}_{\mathrm{II}} \times \boldsymbol{\Phi}_2\cdot\boldsymbol{\omega}_{\mathrm{II}} = \mathbf{T}_2 - \mathbf{T}_H + \mathfrak{L}_2 \times \mathbf{F}_H \tag{1d}$$

where the dots indicate time derivatives with respect to an inertial frame. Because of the constraint imposed by the hinge, the following relations are satisfied

$$\boldsymbol{\varrho}_1 = \boldsymbol{\varrho} + (\mathfrak{L}_1 - \mathfrak{L}_2)\,\frac{m_2}{m} \tag{2a}$$

or

$$\boldsymbol{\varrho}_2 = \boldsymbol{\varrho} + (\mathfrak{L}_2 - \mathfrak{L}_1)\,\frac{m_1}{m}. \tag{2b}$$

Addition of (1a) and (1b) yields the following vector equation which governs the motion of the mass center S_0 :

$$\mathbf{F}_1 + \mathbf{F}_2 = m\ddot{\boldsymbol{\varrho}}. \tag{3}$$

Using (1a), (2a), and (3) we may solve for $\mathbf{F}_H$

$$\mathbf{F}_H = \frac{m_1}{m}\,\mathbf{F}_2 - \frac{m_2}{m}\,\mathbf{F}_1 + \bar{m}(\ddot{\mathfrak{L}}_1 - \ddot{\mathfrak{L}}_2). \tag{4}$$

Inserting (4) in (1c) and (1d) and using the fact that

$$\dot{\mathfrak{L}}_1 = \boldsymbol{\omega}_{\mathrm{I}} \times \mathfrak{L}_1 \quad \text{and} \quad \ddot{\mathfrak{L}}_1 = \dot{\boldsymbol{\omega}}_{\mathrm{I}} \times \mathfrak{L}_1 + \boldsymbol{\omega}_{\mathrm{I}} \times (\boldsymbol{\omega}_{\mathrm{I}} \times \mathfrak{L}_1)\,,$$

etc., equations (1c) and (1d) become

$$\begin{aligned}\boldsymbol{\Phi}_1'\cdot\dot{\boldsymbol{\omega}}_{\mathrm{I}} + \boldsymbol{\omega}_{\mathrm{I}} \times \boldsymbol{\Phi}_1'\cdot\boldsymbol{\omega}_{\mathrm{I}} &= \mathbf{T}_1 + \mathbf{T}_H \\ &+ \mathfrak{L}_1 \times \left\{\frac{m_2}{m}\,\mathbf{F}_1 - \frac{m_1}{m}\,\mathbf{F}_2 + \bar{m}[\boldsymbol{\omega}_{\mathrm{II}} \times (\boldsymbol{\omega}_{\mathrm{II}} \times \mathfrak{L}_2) + \dot{\boldsymbol{\omega}}_{\mathrm{II}} \times \mathfrak{L}_2]\right\}\end{aligned} \tag{5a}$$

$$\begin{aligned}\boldsymbol{\Phi}_2'\cdot\dot{\boldsymbol{\omega}}_{\mathrm{II}} + \boldsymbol{\omega}_{\mathrm{II}} \times \boldsymbol{\Phi}_2'\cdot\boldsymbol{\omega}_{\mathrm{II}} &= \mathbf{T}_2 - \mathbf{T}_H \\ &+ \mathfrak{L}_2 \times \left\{\frac{m_1}{m}\,\mathbf{F}_2 - \frac{m_2}{m}\,\mathbf{F}_1 + \bar{m}[\boldsymbol{\omega}_{\mathrm{I}} \times (\boldsymbol{\omega}_{\mathrm{I}} \times \mathfrak{L}_1) + \dot{\boldsymbol{\omega}}_{\mathrm{I}} \times \mathfrak{L}_1]\right\}\end{aligned} \tag{5b}$$

where $\boldsymbol{\Phi}_i' = \boldsymbol{\Phi}_i + \bar{m}(\ell_i^2\mathbf{I} - \mathfrak{L}_i\mathfrak{L}_i)$, $i = 1, 2$, and $\mathbf{I}$ is the unit dyadic.

III. GRAVITATIONAL FORCE

The earth's gravitational field is taken to be radially symmetric. The gravitational force, $d\mathbf{G}_i$, acting on an infinitesimal mass dm_i at P_i is then

$$d\mathbf{G}_i = -\frac{\mu\, dm_i}{R_i^3}\,\mathbf{R}_i \tag{6}$$

where $\mu = gR_E^2$ with g being the gravitational acceleration at the earth's surface and R_E being the earth's radius. From Fig. 1

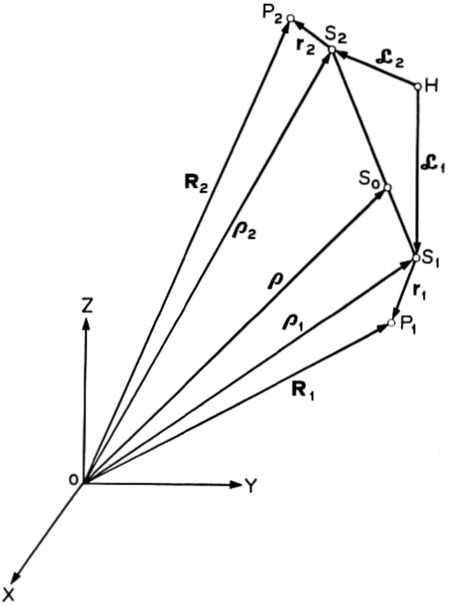

Fig. 1 — Vector displacement diagram of a two-body satellite.

$$
\begin{aligned}
d\mathbf{G}_i &= -\frac{\mu\, dm_i}{\rho_i^3}(\boldsymbol{\varrho}_i + \mathbf{r}_i)\left(1 + \frac{2\boldsymbol{\varrho}_i\cdot\mathbf{r}_i}{\rho_i^2} + \frac{r_i^2}{\rho_i^2}\right)^{-\frac{3}{2}} \\
&= \left[-\frac{\mu\, dm_i}{\rho_i^3}\boldsymbol{\varrho}_i - \frac{\mu\, dm_i}{\rho_i^3}\mathbf{r}_i + \frac{3\mu\boldsymbol{\varrho}_i}{\rho_i^5}(\boldsymbol{\varrho}_i\cdot\mathbf{r}_i)\, dm_i\right]\left[1 + O\left(\frac{l^2}{\rho_i^2}\right)\right] \qquad (7)
\end{aligned}
$$

where the last quantity represents terms of order l^2/ρ_i^2 and higher and l is the maximum linear dimension of the satellite. These higher-order terms are neglected in the analysis. Since S_i is the center of mass of body i

$$\int_{m_i} \mathbf{r}_i\, dm_i = 0.$$

Hence $\mathbf{G}_1$, the gravitational force on body 1, is

$$\mathbf{G}_1 = -\frac{\mu m_1 \boldsymbol{\varrho}_1}{\rho_1^3}\left[1 + O\left(\frac{l^2}{\rho_1^2}\right)\right]$$

or, by (2a)

$$\mathbf{G}_1 = \left[-\frac{\mu m_1 \boldsymbol{\varrho}}{\rho^3} + \frac{\mu \bar{m}}{\rho^3}(\mathcal{L}_2 - \mathcal{L}_1)\cdot(\mathbf{I} - 3\hat{\rho}\hat{\rho})\right]\left[1 + O\left(\frac{l^2}{\rho^2}\right).\right] \qquad (8a)$$

Similarly,

$$\mathbf{G}_2 = \left[-\frac{\mu m_2 \boldsymbol{\varrho}}{\rho^3} - \frac{\mu \bar{m}}{\rho^3} (\mathfrak{L}_2 - \mathfrak{L}_1) \cdot (\mathbf{I} - 3\hat{\rho}\hat{\rho}) \right] \left[1 + O\left(\frac{l^2}{\rho^2} \right) \right] \tag{8b}$$

where the symbol "ˆ" denotes a unit vector. Using (7), the gravitational torque acting on body i about the center of mass is given by

$$\mathbf{T}_{Gi} = \int \mathbf{r}_i \times d\mathbf{G}_i = \frac{3\mu}{\rho^3} \hat{\rho} \times \boldsymbol{\Phi}_i \cdot \hat{\rho} \left[1 + O\left(\frac{l}{\rho} \right) \right], \qquad i = 1, 2. \tag{9}$$

Let $\mathbf{F}_i = \mathbf{F}_i' + \mathbf{G}_i$, $\mathbf{T}_i = \mathbf{T}_i' + \mathbf{T}_{Gi}$, $i = 1, 2$. Substituting in (5*a*,*b*) with the gravitational torques in (9) and the gravitational forces in (8) with terms of $O(l/\rho)$ and $O(l^2/\rho^2)$ neglected, the general equations of rotational motion of two hinged-connected rigid bodies become

$$\begin{aligned} \boldsymbol{\Phi}_1' \cdot \dot{\boldsymbol{\omega}}_{\mathrm{I}} + \boldsymbol{\omega}_{\mathrm{I}} \times \boldsymbol{\Phi}_1' \cdot \boldsymbol{\omega}_{\mathrm{I}} = {} & \frac{3\mu}{\rho^3} \hat{\rho} \times \boldsymbol{\Phi}_1' \cdot \hat{\rho} \\ & + \frac{\mu \bar{m}}{\rho^3} (\mathfrak{L}_1 \times \mathfrak{L}_2 - 3\mathfrak{L}_1 \times \hat{\rho}\hat{\rho} \cdot \mathfrak{L}_2) + \mathbf{T}_1' + \mathbf{T}_H \\ & - \frac{m_1}{m} \mathfrak{L}_1 \times \mathbf{F}_2' + \frac{m_2}{m} \mathfrak{L}_1 \times \mathbf{F}_1' \\ & + \bar{m}(\boldsymbol{\omega}_{\mathrm{II}} \cdot \mathfrak{L}_2 \mathfrak{L}_1 \times \boldsymbol{\omega}_{\mathrm{II}} - \boldsymbol{\omega}_{\mathrm{II}}^2 \mathfrak{L}_1 \times \mathfrak{L}_2 + \mathfrak{L}_1 \cdot \mathfrak{L}_2 \dot{\boldsymbol{\omega}}_{\mathrm{II}} - \mathfrak{L}_2 \mathfrak{L}_1 \cdot \dot{\boldsymbol{\omega}}_{\mathrm{II}}), \end{aligned} \tag{10a}$$

$$\begin{aligned} \boldsymbol{\Phi}_2' \cdot \dot{\boldsymbol{\omega}}_{\mathrm{II}} + \boldsymbol{\omega}_{\mathrm{II}} \times \boldsymbol{\Phi}_2' \cdot \boldsymbol{\omega}_{\mathrm{II}} = {} & \frac{3\mu}{\rho^3} \hat{\rho} \times \boldsymbol{\Phi}_2' \cdot \hat{\rho} \\ & + \frac{\mu \bar{m}}{\rho^3} (\mathfrak{L}_2 \times \mathfrak{L}_1 - 3\mathfrak{L}_2 \times \hat{\rho}\hat{\rho} \cdot \mathfrak{L}_1) + \mathbf{T}_2' - \mathbf{T}_H \\ & - \frac{m_2}{m} \mathfrak{L}_2 \times \mathbf{F}_1' + \frac{m_1}{m} \mathfrak{L}_2 \times \mathbf{F}_2' \\ & + \bar{m}(\boldsymbol{\omega}_{\mathrm{I}} \cdot \mathfrak{L}_1 \mathfrak{L}_2 \times \boldsymbol{\omega}_{\mathrm{I}} - \boldsymbol{\omega}_{\mathrm{I}}^2 \mathfrak{L}_2 \times \mathfrak{L}_1 + \mathfrak{L}_2 \cdot \mathfrak{L}_1 \dot{\boldsymbol{\omega}}_{\mathrm{I}} - \mathfrak{L}_1 \mathfrak{L}_2 \cdot \dot{\boldsymbol{\omega}}_{\mathrm{I}}). \end{aligned} \tag{10b}$$

Note that $\mathbf{T}_1'$ and $\mathbf{T}_2'$ are the resultant torques imposed on body 1 and 2 by some external sources other than gravity. They do not include torques arising from the reaction of one body upon the other. Similarly, $\mathbf{F}_1'$ and $\mathbf{F}_2'$ are the resultant forces on body 1 and 2 due to external sources other than gravity. They do not include the reaction of one body upon the other. Thus both these torques and forces are worked out as though the bodies were not connected. Various environmental disturbances, like solar radiation pressure or interaction of a magnetic moment in the satellite with the geomagnetic field, may be taken into account by assigning appropriate values to $\mathbf{T}_1'$, $\mathbf{T}_2'$, $\mathbf{F}_1'$, and $\mathbf{F}_2'$. This subject is not treated here to conserve space.

From (8a) and (8b) it is seen that

$$\mathbf{G}_1 + \mathbf{G}_2 = -\frac{\mu m \boldsymbol{\varrho}}{\rho^3}. \tag{11a}$$

If the gravitational forces are the only ones in $\mathbf{F}_1$ and $\mathbf{F}_2$, then (3) becomes

$$\ddot{\boldsymbol{\varrho}} = -\frac{\mu \boldsymbol{\varrho}}{\rho^3}. \tag{11b}$$

The solution of this vector equation is an elliptical orbit of S_0, independent of the rotations of the satellite, because of the fact that terms of $O(l^2/\rho^2)$ have now been neglected. As the earth's gravitational field is assumed to be radially symmetric, the orbital plane is fixed in the inertial space.

IV. COORDINATE SYSTEMS

Four reference frames are used to describe the motions of the satellite.

The first frame has its origin at the geocenter O with the Z-axis through the perigee of the orbit and with the Y-axis in the direction of the orbital angular momentum. The X-axis is chosen to form a right-handed set of axes (see Fig. 2). This coordinate system is taken to be inertial.

The second is an earth-pointing frame. It has its origin at the satellite's center of mass, S_0, with the z-axis along OS_0 making an angle ψ with the Z-axis. The y-axis is parallel to the Y-axis, and the x-axis is chosen to form a right-handed system. The relationship between the unit vectors of the coordinate systems O-XYZ and S_0-xyz is

$$\begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} = \begin{pmatrix} C\psi & 0 & -S\psi \\ 0 & 1 & 0 \\ S\psi & 0 & C\psi \end{pmatrix} \begin{pmatrix} \hat{X} \\ \hat{Y} \\ \hat{Z} \end{pmatrix} \tag{12}$$

where S and C are abbreviations of sine and cosine.

The third frame has its origin at S_1 with axes S_1-$x_1y_1z_1$ along the principal axes of inertia of body 1. Euler parameters[2] are employed to describe the motion of S_1-$x_1y_1z_1$ relative to S_0-xyz. The transformation is given by

$$\begin{pmatrix} \hat{x}_1 \\ \hat{y}_1 \\ \hat{z}_1 \end{pmatrix} = (a_{ij}) \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \tag{13a}$$

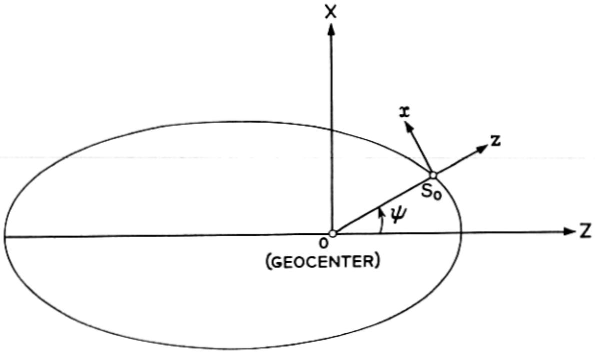

Fig. 2 — Coordinates of the rotating and nonrotating frames.

where

$$(a_{ij}) = \begin{pmatrix} \xi^2 - \eta^2 - \zeta^2 + \chi^2 & 2(\xi\eta + \zeta\chi) & 2(\xi\zeta - \eta\chi) \\ 2(\xi\eta - \zeta\chi) & -\xi^2 + \eta^2 - \zeta^2 + \chi^2 & 2(\xi\chi + \eta\zeta) \\ 2(\xi\zeta + \eta\chi) & 2(-\xi\chi + \eta\zeta) & -\xi^2 - \eta^2 + \zeta^2 + \chi^2 \end{pmatrix} \tag{13b}$$

$i, j = 1, 2, 3$ representing rows and columns respectively, and

$$\xi^2 + \eta^2 + \zeta^2 + \chi^2 = 1. \tag{13c}$$

The fourth frame has an origin at S_2 with axes $S_2\text{-}x_2y_2z_2$ along the principal axes of inertia of body 2. If a universal joint is used, the relative rotation of the second body can be completely specified with only two angles, namely α, the rotation of the journal in body 1, and β, the rotation of the journal of body 2. When these two journals are directed respectively along $\hat{x}_1$ and $\hat{y}_2$, then the transformation from $S_1\text{-}x_1y_1z_1$ to $S_2\text{-}x_2y_2z_2$ is given by

$$\begin{pmatrix} \hat{x}_2 \\ \hat{y}_2 \\ \hat{z}_2 \end{pmatrix} = (b_{ij}) \begin{pmatrix} \hat{x}_1 \\ \hat{y}_1 \\ \hat{z}_1 \end{pmatrix} \tag{14a}$$

where

$$(b_{ij}) = \begin{pmatrix} C\beta & S\alpha\, S\beta & -C\alpha\, S\beta \\ 0 & C\alpha & S\alpha \\ S\beta & -S\alpha\, C\beta & C\alpha\, C\beta \end{pmatrix}. \tag{14b}$$

The constraint equation $\hat{x}_1 \cdot \hat{y}_2 = 0$ is automatically satisfied by the introduction of the two coordinate parameters α and β. The angular velocities of the two bodies are

$$\boldsymbol{\omega}_\mathrm{I} = \dot{\psi}\hat{y} + \lambda_1\hat{x}_1 + \lambda_2\hat{y}_1 + \lambda_3\hat{z}_1 \tag{15a}$$

where

$$\lambda_1 = 2(\chi\dot{\xi} + \zeta\dot{\eta} - \eta\dot{\zeta} - \xi\dot{\chi}) \tag{15b}$$

$$\lambda_2 = 2(-\zeta\dot{\xi} + \chi\dot{\eta} + \xi\dot{\zeta} - \eta\dot{\chi}) \tag{15c}$$

$$\lambda_3 = 2(\eta\dot{\xi} - \xi\dot{\eta} + \chi\dot{\zeta} - \zeta\dot{\chi}) \tag{15d}$$

and

$$\boldsymbol{\omega}_{\mathrm{II}} = \boldsymbol{\omega}_{\mathrm{I}} + \dot{\alpha}\hat{x}_1 + \dot{\beta}\hat{y}_2 . \tag{15e}$$

V. SPECIALIZED EQUATIONS OF MOTION

Let us specialize our satellite so that $\mathfrak{L}_2 = 0$ and $\mathfrak{L}_1 = -\ell_1\hat{z}_1$. We assume gravity to be the only external force, i.e., $\mathbf{T}_1'$, $\mathbf{T}_2'$, $\mathbf{F}_1'$ and $\mathbf{F}_2'$ in (10) are taken to be zero. Then equations (10) are equivalent to those derived from two bodies connected at their centers of mass except that the inertia dyadic $\mathbf{\Phi}_1$ is replaced by $\mathbf{\Phi}_1'$ defined in (5) ($\mathbf{\Phi}_2' = \mathbf{\Phi}_2$ as $\ell_2 = 0$). The two bodies are connected by a universal joint, which is characterized by an interposed weightless body, having two perpendicular journals as previously described. The torque $\mathbf{T}_H$, transmitted through the universal joint, consists of the constraint torque $\mathbf{T}_c$, the elastic restoring torque $\mathbf{T}_r$, and the dissipative torque $\mathbf{T}_d$. The components of the latter two along the journals x_1 and y_2 are specified by subscripts 1 and 2 respectively. Hence $\mathbf{T}_H$ can be written as

$$\mathbf{T}_H = T_c\hat{x}_1 \times \hat{y}_2 + (T_{r1} + T_{d1})\hat{x}_1 + (T_{r2} + T_{d2})\hat{y}_2 . \tag{16}$$

Let

$$I_1 = \mathbf{\Phi}_1' \cdot \hat{x}_1 \tag{17a}$$

$$I_2 = \mathbf{\Phi}_1' \cdot \hat{y}_1 \tag{17b}$$

$$I_3 = \mathbf{\Phi}_1' \cdot \hat{z}_1 \tag{17c}$$

$$I_4 = \mathbf{\Phi}_2 \cdot \hat{x}_2 \tag{17d}$$

$$I_5 = \mathbf{\Phi}_2 \cdot \hat{y}_2 \tag{17e}$$

$$I_6 = \mathbf{\Phi}_2 \cdot \hat{z}_2 \tag{17f}$$

$$\omega_i(i = 1, 2, 3) = \text{components of } \boldsymbol{\omega}_{\mathrm{I}} \text{ along } \hat{x}_1, \hat{y}_1, \hat{z}_1 \tag{17g}$$

$$\omega_i(i = 4, 5, 6) = \text{components of } \boldsymbol{\omega}_{\mathrm{II}} \text{ along } \hat{x}_2, \hat{y}_2, \hat{z}_2 . \tag{17h}$$

From the orbit equation (11b), the following relations can be derived

$$\dot{\psi} = \frac{\Omega}{(1 - \epsilon^2)^{\frac{3}{2}}} (1 + \epsilon C\psi)^2 \tag{18}$$

$$G = \frac{3\mu}{\rho^3} = \frac{3\Omega^2}{(1 - \epsilon^2)^3} (1 + \epsilon C\psi)^3 \tag{19}$$

where

$$\epsilon = \text{eccentricity of the orbit}$$

$$\Omega = 2\pi \text{ divided by the orbital period.}$$

Euler's equations of motion (10), simplified for the specialized satellite, are written out as

$$I_1\dot{\omega}_1 = (I_2 - I_3)(\omega_2\omega_3 - Gn_2n_3) + T_{r1} + T_{d1} \tag{20a}$$

$$I_2\dot{\omega}_2 = (I_3 - I_1)(\omega_3\omega_1 - Gn_3n_1) + (T_{r2} + T_{d2})C\alpha - T_cS\alpha \tag{20b}$$

$$I_3\dot{\omega}_3 = (I_1 - I_2)(\omega_1\omega_2 - Gn_1n_2) + (T_{r2} + T_{d2})S\alpha + T_cC\alpha \tag{20c}$$

$$I_4\dot{\omega}_4 = (I_5 - I_6)(\omega_5\omega_6 - Gn_5n_6) - (T_{r1} + T_{d1})C\beta + T_cS\beta \tag{20d}$$

$$I_5\dot{\omega}_5 = (I_6 - I_4)(\omega_6\omega_4 - Gn_6n_4) - T_{r2} - T_{d2} \tag{20e}$$

$$I_6\dot{\omega}_6 = (I_4 - I_5)(\omega_4\omega_5 - Gn_4n_5) - (T_{r1} + T_{d1})S\beta - T_cC\beta \tag{20f}$$

where

$$n_i = a_{i3} \qquad \text{(see 13b)} \qquad i = 1, 2, 3 \tag{20g}$$

$$n_{i+3} = \sum_{k=1}^{3} b_{ik}a_{k3} \qquad \text{(see 14b)} \qquad i = 1, 2, 3. \tag{20h}$$

Because of the constraint $\hat{x} \cdot \hat{y} = 0$, a relation must exist among the six ω_i's. Such a relation, i.e., $(\omega_{\text{I}} - \omega_{\text{II}}) \cdot (\hat{x}_1 \times \hat{y}_2) = 0$, can be obtained from (15e). This yields the following relationship:

$$\omega_2S\alpha - \omega_3C\alpha - \omega_4S\beta + \omega_6C\beta = 0. \tag{21}$$

If (21) is differentiated and equations (20) are substituted, the unknown T_c is found to be

$$T_c = \left(\frac{S^2\alpha}{I_2} + \frac{C^2\alpha}{I_3} + \frac{S^2\beta}{I_4} + \frac{C^2\beta}{I_6}\right)^{-1}$$

$$\left\{\frac{S\alpha}{I_2}[(I_3 - I_1)(\omega_1\omega_3 - Gn_1n_3) + (T_{r2} + T_{d2})\,C\alpha]\right.$$

$$- \frac{C\alpha}{I_3}[(I_1 - I_2)(\omega_1\omega_2 - Gn_1n_2) + (T_{r2} + T_{d2})\,S\alpha]$$

$$- \frac{S\beta}{I_4}[(I_5 - I_6)(\omega_5\omega_6 - Gn_5n_6) - (T_{r1} + T_{d1})\,C\beta]$$

$$+ \frac{C\beta}{I_6} [(I_4 - I_5)(\omega_4\omega_5 - Gn_4n_5) - (T_{r1} + T_{d1}) S\beta]$$

$$\left. + \dot{\alpha}(\omega_2 C\alpha + \omega_3 S\alpha) - \dot{\beta}(\omega_4 C\beta + \omega_6 S\beta) \right\}. \qquad (22)$$

Equations (20) could be considered as a system of six second-order equations in six unknowns ξ, η, ζ, χ, α, β, while ψ is determined from (18) and the ω_i's from (15). For computation purposes it is convenient to also leave the ω_i's as dependent variables. Equations (15) give

$$\dot{\xi} = \tfrac{1}{2}(\chi\lambda_1 - \zeta\lambda_2 + \eta\lambda_3) \qquad (23a)$$

$$\dot{\eta} = \tfrac{1}{2}(\zeta\lambda_1 + \chi\lambda_2 - \xi\lambda_3) \qquad (23b)$$

$$\dot{\xi} = \tfrac{1}{2}(-\eta\lambda_1 + \xi\lambda_2 + \chi\lambda_3) \qquad (23c)$$

$$\dot{\chi} = \tfrac{1}{2}(-\xi\lambda_1 - \eta\lambda_2 - \zeta\lambda_3) \qquad (23d)$$

$$\dot{\alpha} = -\omega_1 + \omega_4 C\beta + \omega_6 S\beta \qquad (23e)$$

$$\dot{\beta} = -\omega_2 C\alpha - \omega_3 S\alpha + \omega_5 \qquad (23f)$$

$$\lambda_i \equiv \omega_i - a_{i2}\dot{\psi}, \qquad i = 1, 2, 3. \qquad (23g)$$

There are now 12 first-order equations, (20 a-f) and (23a-f), in the unknowns ξ, η, ζ, χ, α, β, ω_1, ω_2, ω_3, ω_4, ω_5, and ω_6. If Euler angles had been used instead of Euler parameters, there would be certain positions of the body for which the derivatives of the angles have a singularity. However, no singularities occur when Euler parameters are used, as can be seen from (23). It should also be noticed from (13b) that the matrix fixing the body position is not changed if the coordinates (ξ, η, ζ, χ) are replaced by ($-\xi$, $-\eta$, $-\zeta$, $-\chi$).

VI. DISSIPATIVE AND ELASTIC TORQUES IN THE UNIVERSAL JOINT

To completely define the problem it is necessary to specify the elastic and dissipative torques $\mathbf{T}_r$ and $\mathbf{T}_d$.

6.1 *Damping Torques*

Two types of damping torques are considered here. The first is viscous damping of the linear velocity type; the torque on body 1 has two components

$$\mathbf{T}_{d1} = C_1\dot{\alpha}\hat{x}_1 \qquad (24a)$$

$$\mathbf{T}_{d2} = C_2\dot{\beta}\hat{y}_2 \qquad (24b)$$

where C_1 and C_2 are viscous damping coefficients.

The second is of magnetic hysteresis type. The damping is furnished by hysteresis losses produced by the relative motion of a permanent magnet and a permeable material. The torque in the x_1-direction might be approximately expressed by the following process. If $\dot{\alpha} > 0$, the torque would be represented in region I in Fig. 3 by

$$T_{d1} = T_{d1}^* + \bar{T}_{d1} \frac{\alpha - \alpha^*}{\bar{\alpha}} \tag{25}$$

as long as $| T_{d1} | < \bar{T}_{d1}$ where $\bar{\alpha}$, $\bar{T}_{d1}$ are constants and α^*, T_{d1}^* are the values of α, T_{d1} when $\dot{\alpha}$ last changed sign. After $| T_{d1} |$ reaches $\bar{T}_{d1}$ then T_{d1} remains at $\bar{T}_{d1}$ as long as $\dot{\alpha}$ does not change sign. This is represented as region II in Fig. 3. If $\dot{\alpha}$ changes sign, then (25) applies and the process is repeated. This is represented by region III of Fig. 3. T_{d2} is defined by replacing α by β and subscript 1 by 2 in (25). According to this idealized hysteresis, no energy is dissipated in region III. In an actual device, energy would also be dissipated in this region because of minor hysteresis loops. The chief advantage of magnetic hysteresis damping is that it is amplitude dependent instead of velocity dependent, since the librational frequency, which is of the order of the orbital frequency, is too low to make the velocity damping effective. Other merits of the magnetic hysteresis damper will be stated in the descriptions of a practical design for a numerical computation.

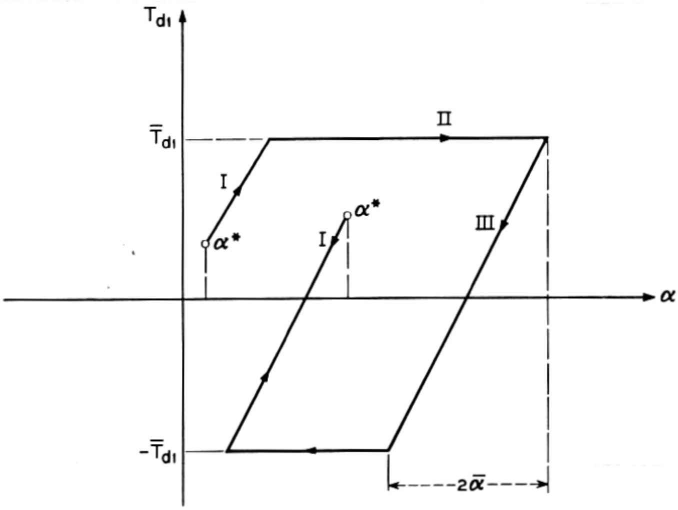

Fig. 3 — Magnetic hysteresis damping torque produced by a magnetic device on the x_1-journal.

6.2 *Elastic Torques*

It is assumed that each journal is furnished with a linearly elastic restoring torque produced, for example, by the torsion of a wire. The torque acting on body 1 is given by

$$\mathbf{T}_{r1} = k_1 \alpha \hat{x}_1 \tag{26a}$$

$$\mathbf{T}_{r2} = k_2 \beta \hat{y}_2 . \tag{26b}$$

where k_1 and k_2 are spring constants.

VII. MISCELLANEOUS TORQUES

Many other torques such as those due to interaction of the satellite's magnetic moments with the geomagnetic field, solar radiation, self-gravitation between two bodies, and plasma effects will act as forcing terms in the equations of motion. By proper design, these torques can be made small compared to the gravitational torque. However, since the gravitational torque varies inversely as the cube of the geocentric distance, it may not necessarily dominate in the orientation of satellites in very high orbits. Also, in very low orbits, aerodynamic drag may be big enough to upset the orientation. If long rods are used with weights on the ends, the gravitational torque can be made to dominate for a certain range in altitude.

VIII. EQUILIBRIUM AND STABILITY

Let us consider only the equilibrium position

$$(\xi,\eta,\zeta,\chi,\alpha,\beta) = (0,0,0,1,0,0)$$

in which the x_1 , y_1 , z_1 axes are lined up with the x,y,z axes. For viscous damping, the stability criteria for the position (0,0,0,1,0,0) can be found by linearizing the equations of motion about this position. The same stability criteria are obtained for equilibrium positions found by rotations of 180° around the x, y, and z axes, i.e., (1,0,0,0,0,0), (0,1,0,0,0,0), (0,0,1,0,0,0). For hysteresis damping, there will be an infinite number of stable equilibrium positions. All of these can, however, be made sufficiently close together to either one of the above four equilibrium positions, thus maintaining an axis in the satellite nearly in line with the local vertical.

From the definition of Euler parameters, the infinitesimal angles of rotation about the x_1 , y_1 , and z_1 axes are $\xi_1 = 2\xi$, $\eta_1 = 2\eta$, $\zeta_1 = 2\zeta$, defined as the roll, pitch and yaw angles. If ξ_2 , η_2 , ζ_2 are the infinitesimal angles that the principal axes of body 2 make with respect to the rotating coordinate system S_0-xyz, and α and β are small, then

$$\xi_2 = \alpha + \xi_1 \tag{27a}$$

$$\eta_2 = \beta + \eta_1 \tag{27b}$$

$$\zeta_2 = \zeta_1 . \tag{27c}$$

In the linearization process, we take the eccentricity of the orbit, ϵ' to be small in order to insure the realization of the infinitesimal angles' This is necessary in view of the well known result of the satellite pitch motion that the angular excursion produced by the eccentricity is of the same order of magnitude as the eccentricity itself. From (18) ψ becomes, with zero phase angle,

$$\psi = \Omega t + 2\epsilon S\Omega t + 0(\epsilon^2). \tag{28}$$

To linearize the general equations of motion given by (10), let us assume viscous damping as expressed in (24) and linear restoring torques as given in (26). The perturbing torques and forces, $\mathbf{T}_i'$ and $\mathbf{F}_i'$ ($i = 1, 2$), are neglected. Also, let $\mathfrak{L}_1 = -\ell_1\hat{z}_1$ and $\mathfrak{L}_2 = \ell_2\hat{z}_2$. Then, equations (10) are linearized to the following:

$$\ddot{\eta}_1 + L_1\ddot{\eta}_2 + C_1'(\dot{\eta}_1 - \dot{\eta}_2) + d_1\eta_1 - k_1'\eta_2 = 2\epsilon\Omega^2(1 + L_1)S\Omega t \tag{29a}$$

$$\ddot{\eta}_2 + L_2\ddot{\eta}_1 + C_2'(\dot{\eta}_2 - \dot{\eta}_1) + d_2\eta_2 - k_2'\eta_1 = 2\epsilon\Omega^2(1 + L_2)S\Omega t \tag{29b}$$

$$\ddot{\xi}_1 + N_1\ddot{\xi}_2 + C_1''(\dot{\xi}_1 - \dot{\xi}_2) + q_1\Omega\dot{\zeta} + u_1\xi_1 - \bar{k}_1\xi_2 = 0 \tag{29c}$$

$$\ddot{\xi}_2 + N_2\ddot{\xi}_1 + C_2''(\dot{\xi}_2 - \dot{\xi}_1) + q_2\Omega\dot{\zeta} + u_2\xi_2 - \bar{k}_2\xi_1 = 0 \tag{29d}$$

$$\ddot{\zeta} + (1 - f_1 - f_2)\Omega^2\zeta - \Omega f_1\dot{\xi}_1 - \Omega f_2\dot{\xi}_2 = 0 \tag{29e}$$

where

$$L_1 = \bar{m}\ell_1\ell_2/I_2, \qquad L_2 = \bar{m}\ell_1\ell_2/I_5$$

$$C_1' = C_2/I_2, \qquad C_2' = C_2/I_5$$

$$k_1' = k_2/I_2, \qquad k_2' = k_2/I_5$$

$$d_1 = 3\Omega^2(I_1 - I_3)/I_2 + 3\Omega^2L_1 + k_1'$$

$$d_2 = 3\Omega^2(I_4 - I_6)/I_5 + 3\Omega^2L_2 + k_2'$$

$$N_1 = \bar{m}\ell_1\ell_2/I_1, \qquad N_2 = \bar{m}\ell_1\ell_2/I_4$$

$$C_1'' = C_1/I_1, \qquad C_2'' = C_1/I_4$$

$$q_1 = (I_1 + I_3 - I_2)/I_1$$

$$q_2 = (I_4 + I_6 - I_5)/I_4$$

$$u_1 = 4\Omega^2(1 - q_1 + 3N_1/4) + k_1/I_1$$

$$u_2 = 4\Omega^2(1 - q_2 + 3N_2/4) + k_1/I_4$$

$$\bar{k}_1 = k_1/I_1 - \Omega^2 N_1, \qquad \bar{k}_2 = k_1/I_4 - \Omega^2 N_2$$

$$f_1 = (I_1 + I_3 - I_2)/(I_3 + I_6), \qquad f_2 = (I_4 + I_6 - I_5)/(I_3 + I_6).$$

It should be noticed that the pitch equations (29a,b) do not depend on ξ_1, ξ_2, ζ and are decoupled from the roll and yaw equations (29c,d,e). The eccentricity enters as the amplitude of a forcing term in pitch but not in roll and yaw. The transient part of the pitch libration can be solved from (29a,b), excluding the forcing terms, by substituting with

$$\eta_i = B_i e^{st}, \qquad i = 1,2.$$

The resulting characteristic equation in s is then

$$(1 - L_1L_2)s^4 + (C_1' + C_2' + C_1'L_2 + C_2'L_1)s^3 + (d_1 + d_2 + k_1'L_2 + k_2'L_1)s^2 + (d_1C_2' + d_2C_1' - C_1'k_2' - C_2'k_1')s + (d_1d_2 - k_1'k_2') = 0. \quad (30)$$

The pitch motion is damped about (0,0,0,1,0,0) if and only if the Routh-Hurwitz conditions[3] are satisfied. This insures that the real parts of the roots of (30), representing the damping constants for the two principal modes, are negative. These give

$$I_x > I_z \quad (31a)$$

$$k_2 > -3\Omega^2 \frac{(I_1 - I_3 + \bar{m}\ell_1\ell_2)}{I_x - I_z}(I_4 - I_6 + \bar{m}\ell_1\ell_2) \quad (31b)$$

$$\frac{I_1 - I_3 + \bar{m}\ell_1\ell_2}{I_2 + \bar{m}\ell_1\ell_2} \neq \frac{I_4 - I_6 + \bar{m}\ell_1\ell_2}{I_5 + \bar{m}\ell_1\ell_2} \quad (31c)$$

where

$$I_x = I_1 + I_4 + 2\bar{m}\ell_1\ell_2$$

$$I_y = I_2 + I_5 + 2\bar{m}\ell_1\ell_2$$

$$I_z = I_3 + I_6 .$$

I_x, I_y, I_z represent the moments of inertia of the composite body about S_0. Condition (31a) is the same as that of a single rigid body. Condition (31b) states that k_2 must be larger than a certain critical value if one body is unstable (e.g., $I_4 - I_6 + \bar{m}\ell_1\ell_2 < 0$). This value is zero if both bodies are stable by themselves. It can be shown that there cannot exist a cocked equilibrium position in pitch if the parameters are such as to make the position (0,0,0,1,0,0) stable. Condition (31c) implies that there exists an undamped motion if the equality sign holds. This rigid body motion has a frequency ω_r, given by

$$\omega_r^2 = 3\Omega^2 \frac{(I_1 - I_3 + \bar{m}\ell_1\ell_2)}{I_2 + \bar{m}\ell_1\ell_2}. \tag{32}$$

The roll and yaw equations (29c,d,e) are all coupled. This justifies the use of a damper only for roll. Up to first-order terms, there are no forcing terms due to eccentricity. The characteristic equation is

$$b_6 s^6 + b_5 s^5 + b_4 s^4 + b_3 s^3 + b_2 s^2 + b_1 s + b_0 = 0 \tag{33}$$

where

$$b_0 = \Omega^2(1 - f_1 - f_2)(u_1 u_2 - \bar{k}_1 \bar{k}_2)$$

$$b_1 = \Omega^2(1 - f_1 - f_2)(C_2'' u_1 + C_1'' u_2 - \bar{k}_1 C_2'' - \bar{k}_2 C_1'')$$

$$b_2 = \Omega^2(1 - f_1 - f_2)(u_1 + u_2 + N_1 \bar{k}_2 + N_2 \bar{k}_1) + u_1 u_2 - \bar{k}_1 \bar{k}_2$$
$$+ \Omega^2(f_1 q_2 \bar{k}_1 + f_2 q_1 \bar{k}_2 + f_1 q_1 u_2 + f_2 q_2 u_1)$$

$$b_3 = \Omega^2(1 - f_1 - f_2)(C_1'' + C_2'' + C_2'' N_1 + C_1'' N_2)$$
$$+ C_1''(u_2 + \Omega^2 f_1 q_2 + \Omega^2 f_2 q_2 - \bar{k}_2)$$
$$+ C_2''(u_1 + \Omega^2 f_2 q_1 + \Omega^2 f_1 q_1 - \bar{k}_1)$$

$$b_4 = \Omega^2(1 - f_1 - f_2)(1 - N_1 N_2) + u_1 + u_2 + N_1 \bar{k}_2 + N_2 \bar{k}_1$$
$$+ \Omega^2(f_1 q_1 + f_2 q_2 - f_1 q_2 N_1 - f_2 q_1 N_2)$$

$$b_5 = C_1''(1 + N_2) + C_2''(1 + N_1)$$

$$b_6 = 1 - N_1 N_2 .$$

The Routh-Hurwitz stability criteria are

$$b_0 > 0, \qquad b_1 > 0, \qquad \begin{vmatrix} b_1 & b_0 \\ b_3 & b_2 \end{vmatrix} > 0,$$

$$\begin{vmatrix} b_1 & b_0 & 0 \\ b_3 & b_2 & b_1 \\ b_5 & b_4 & b_3 \end{vmatrix} > 0, \qquad \begin{vmatrix} b_1 & b_0 & 0 & 0 \\ b_3 & b_2 & b_1 & b_0 \\ b_5 & b_4 & b_3 & b_2 \\ 0 & b_6 & b_5 & b_4 \end{vmatrix} > 0,$$

$$\begin{vmatrix} b_1 & b_0 & 0 & 0 & 0 \\ b_3 & b_2 & b_1 & b_0 & 0 \\ b_5 & b_4 & b_3 & b_2 & b_1 \\ 0 & b_6 & b_5 & b_4 & b_3 \\ 0 & 0 & 0 & b_6 & b_5 \end{vmatrix} > 0, \qquad b_6 > 0.$$

If these are satisfied, there will be three more modes of damped librations. Due to the coupling between the roll and yaw librations, the yaw libration can be damped out by the roll damping, as can be observed from (29c,d,e), although no yaw damping mechanism is provided in the present scheme. Hence, all modes can be damped out and the satellite will oscillate with some steady-state amplitude about an equilibrium

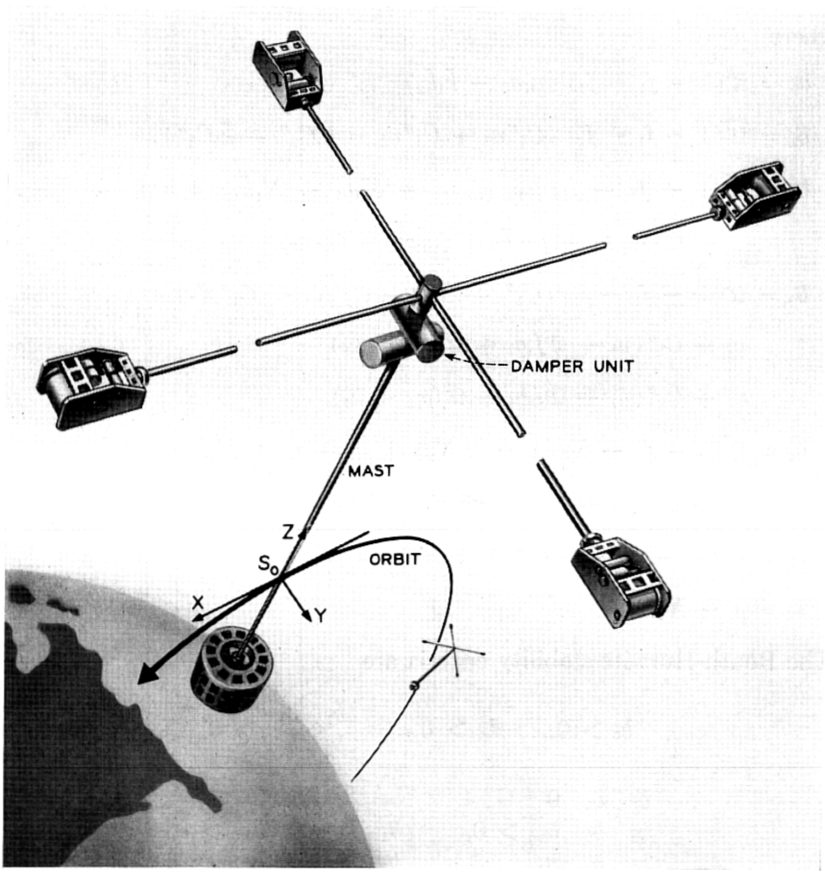

Fig. 4 — Gravitationally oriented two-body satellite with extensible rods.

position. Some of these conditions are too complicated to give any physical insight. However, some are quite simple and are given below.

Since the parts of b_0, b_2 and b_4 which involve k_1 are $(k_1/C_1)b_1$, $(k_1/C_1)b_3$, $(k_1/C_1)b_5$ respectively, multiplying the odd columns of the Hurwitz determinants by k_1/C_1 and adding to adjacent columns will eliminate the k_1 terms. Hence the only condition on k_1 is $b_0 > 0$, i.e.,

$$k_1 > \Omega^2 \frac{[4(I_2 - I_3 + \frac{3}{4}\,\bar{m}\ell_1\ell_2)(I_5 - I_6 + \frac{3}{4}\,\bar{m}\ell_1\ell_2) - \frac{1}{4}\,\bar{m}\ell_1^{\,2}\ell_2^{\,2}]}{I_z - I_y} \tag{34}$$

As k_1 and k_2 approach infinity, the satellite becomes one rigid body. Since the stability conditions are not changed by an increase of k_1 (and k_2),

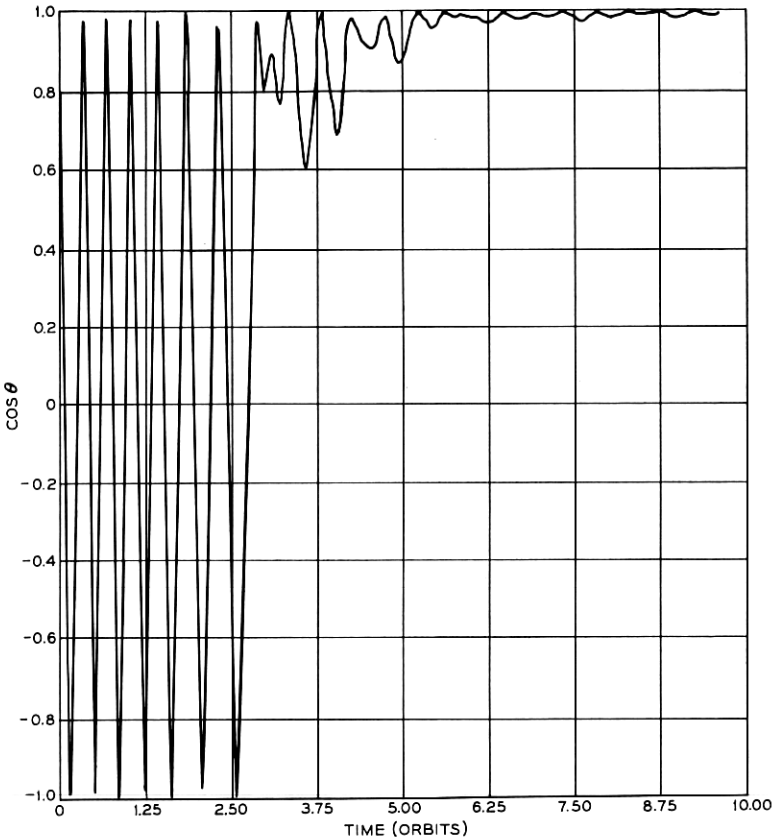

Fig. 5 — Angular variation between the z_1-axis of the satellite and the local vertical for a hysteresis damper, $\cos\theta$.

it appears that the single rigid body criteria for roll and yaw stability are necessary. These are

$$P_x P_z > 0 \tag{35a}$$

$$1 + 3P_x + P_x P_z > 4\sqrt{P_x P_z} \tag{35b}$$

where

$$P_x = \frac{I_y - I_z}{I_x}, \qquad P_z = \frac{I_y - I_x}{I_z}$$

and I_x, I_y, and I_z are given in (31). Condition (35a) can be verified from the inequality $b_1 > 0$. Other necessary conditions in the case of $\ell_2 = 0$ are found from the third-order Hurwitz determinant to be

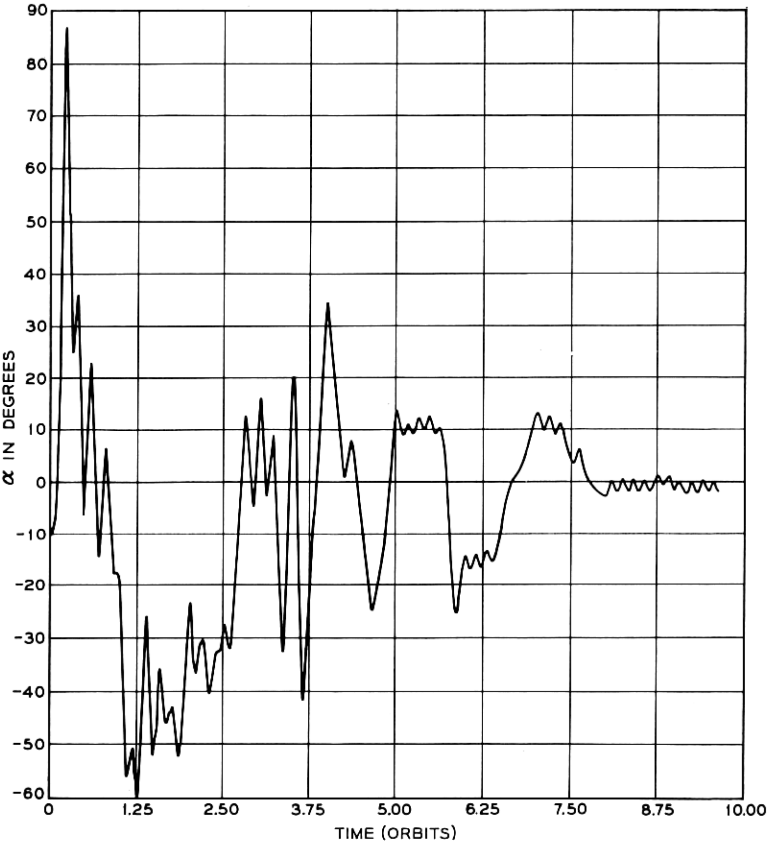

Fig. 6 — Relative angle about the x_1-journal for a hysteresis damper, α.

$$(I_y - I_x)(I_y - \tfrac{3}{4}I_z) > 0 \tag{35c}$$

and

$$\frac{I_5 - I_6}{I_4} \neq \frac{I_2 - I_3}{I_1}. \tag{35d}$$

IX. BISTABILITY

The satellite is in a stable equilibrium position if the z_1-axis is in line with the local vertical (i.e., the z-axis) pointing in either direction. If a directional device such as an antenna or a camera is used along the negative z_1-axis, it may point at or away from the earth. The equi-

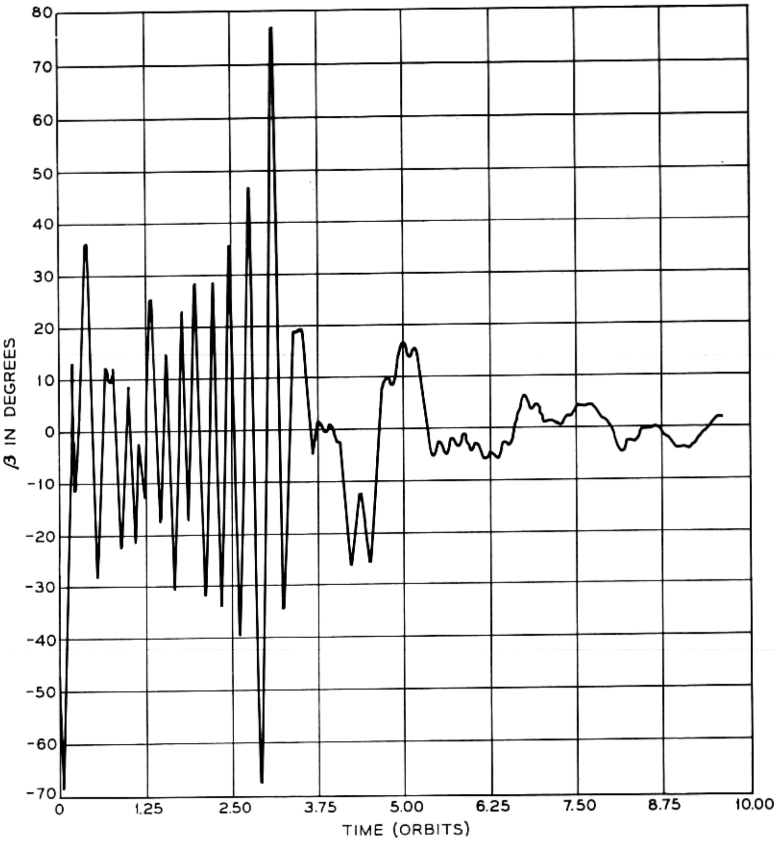

Fig. 7— Relative angle about the y_2-journal for a hysteresis damper, β.

librium positions (0,0,0,1,0,0) and (0,0,1,0,0,0) correspond to the device pointing toward the earth, whereas (1,0,0,0,0,0) and (0,1,0,0,0,0) correspond to the device pointing away from the earth. In the latter case an inertia wheel in the satellite can be activated with a predetermined number of turns, and the satellite can be rotated 180 degrees so that the device will be earth-pointing. The equations governing this turning are given by (10), where the applied torque on body 1 is approximately

$$\mathbf{T}_1' = -\frac{d}{dt}\left[J_m(C\delta\hat{x}_1 + S\delta\hat{y}_1)\right] \tag{36}$$

where J_m is the angular momentum of the inertia wheel and δ is the angle between the x_1-axis and the axis of the inertia wheel. Another scheme

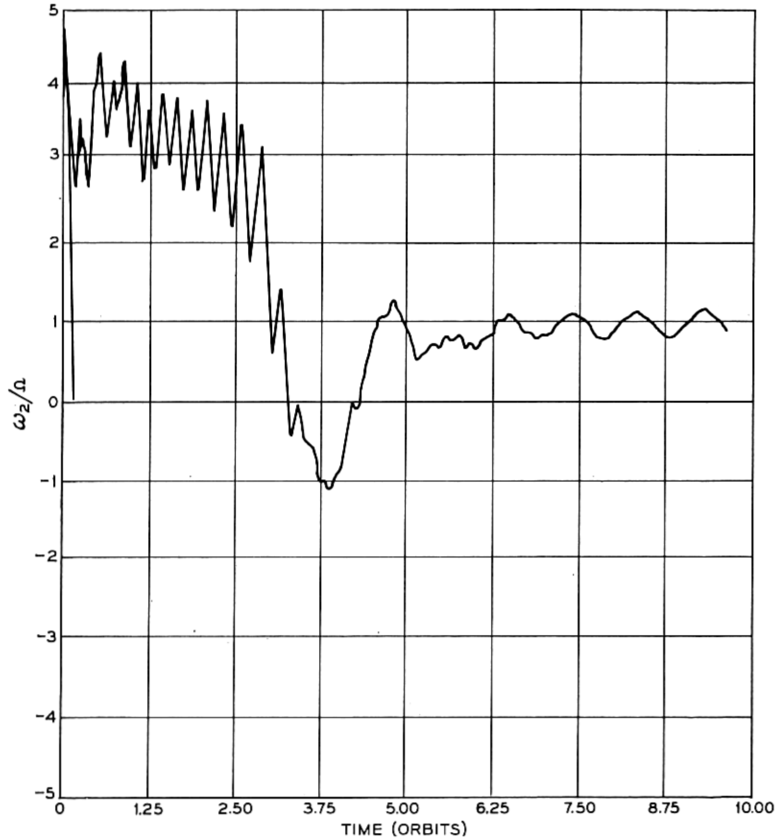

Fig. 8 — Component of angular velocity of the satellite along the y_1-axis for a hysteresis damper, ω_2/Ω.

would be to use two devices, one on each side of the satellite, directed along the positive and the negative direction of the z_1-axis respectively. Only the one that is earth-pointing would be activated.

X. NUMERICAL RESULTS OF A PRACTICAL SCHEME

A practical scheme, as shown in Fig. 4, is suggested here for a communications satellite. The particular construction, employing extensible rods and tip masses, is to effect large moments of inertia so that the gravitational torque will dominate over all disturbing torques. Body 1 of the satellite, which consists of the satellite's main structure (with directional antennas) and a mast rod, is to be earth-pointing. Body 2, being an auxiliary body for attitude-control purpose only, is constructed of two rods and is in an unstable position with respect to the local vertical. These rods are extended, upon ejection from the launching vehicle's

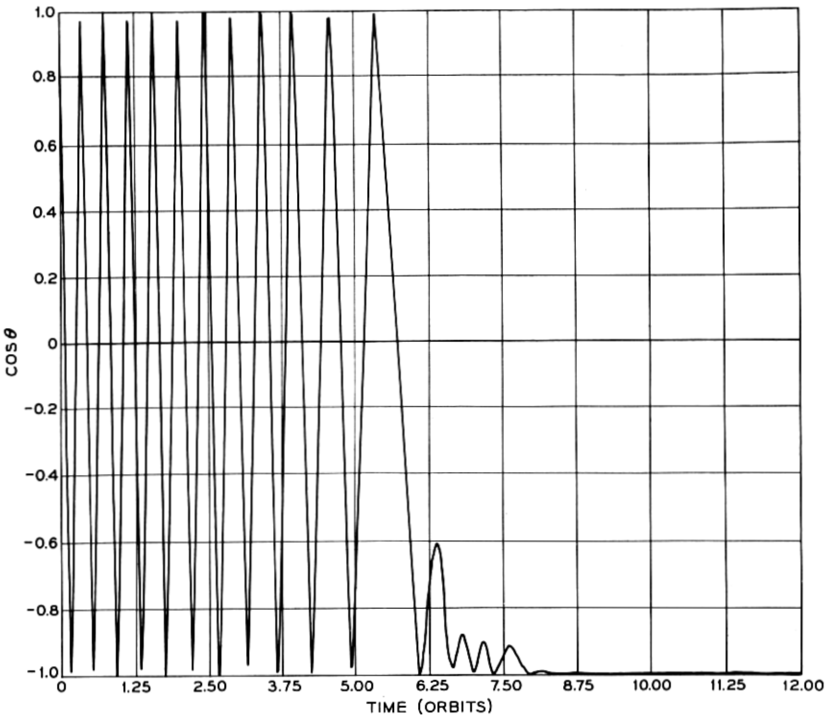

Fig. 9 — Angular variation between the z_1-axis of the satellite and the local vertical for a viscous damper, cos θ.

final stage, by unrolling from sheet metal drums. The universal joint employs torsion wires to produce elastic restoring torques and provides hysteresis damping by relative displacement between magnets and a permeable material. (See Fig. 5 of companion paper.[4])

The advantages of magnetic hysteresis damping are that it is amplitude dependent, insensitive to temperature variation, involves no sliding parts and requires little weight. Coulomb friction damping, while also amplitude dependent, is less desirable because of possible cold welding of sliding parts in the high vacuum of space. Velocity-dependent damping by employing viscous fluids is believed to provide lower damping for a given weight, and the viscous fluids involve questions of temperature sensitivity.

All the parameters are chosen based on the adjusted moment of inertia, I_1, of body 1 subject to stability criteria and other necessary considerations. The stability criteria (31b) and (34) specifying the critical values

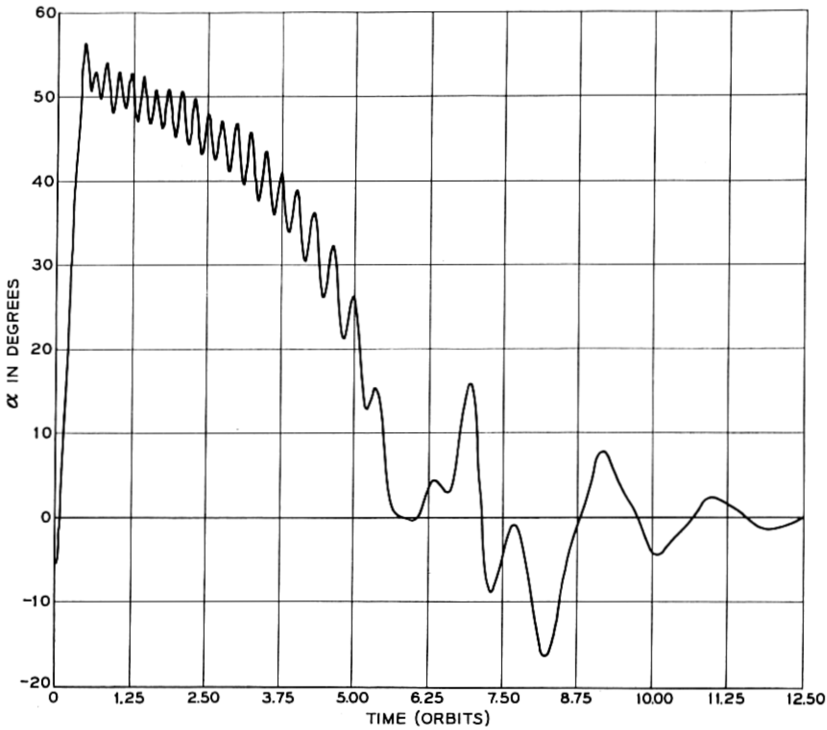

Fig. 10 — Relative angle about the x_1-journal for a viscous damper, α.

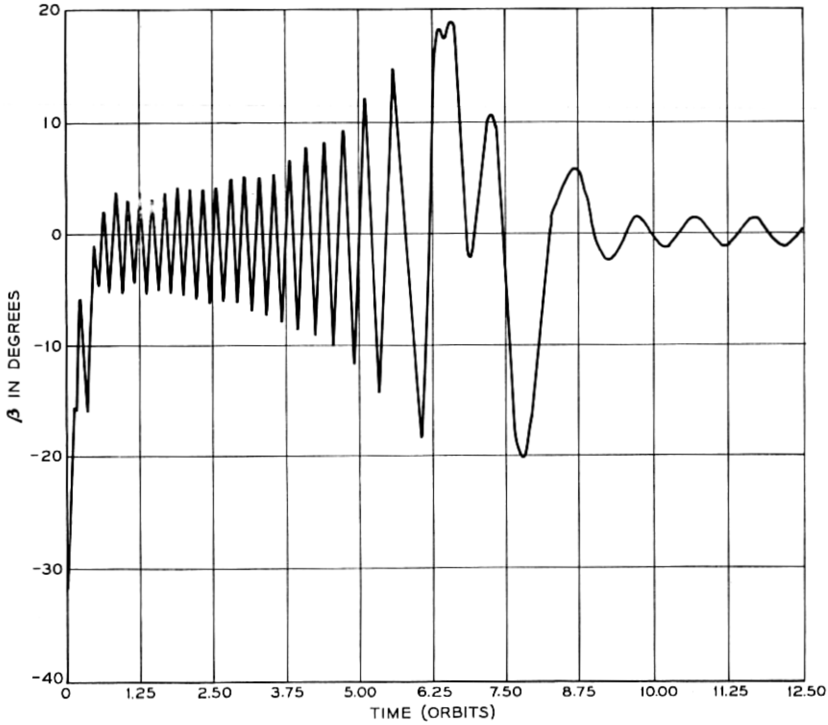

Fig. 11 — Relative angle about the y_2-journal for a viscous damper, β.

of k_2 and k_1, respectively, which are derived from viscous damping, are found to apply approximately also in the case of hysteresis damping. These parameters are: $I_i/I_1 = 1.00, 0.003, 0.159, 0.381, 0.540$ $(i = 2, \cdots, 6)$; $k_i/I_1\Omega^2 = 1.131, 2.238$ $(i = 1,2)$; for a hysteresis damper: $\bar{T}_{di}/I_1\Omega^2 = 0.159, 0.216$ $(i = 1,2)$, $\bar{\alpha} = \bar{\beta} = 2°$; for a viscous damper: $C_i/I_1\Omega = 0.870, 1.281$ $(i = 1,2)$. With the above value of the viscous constant C_2, the amplitude of the lower mode of pitch libration can be reduced according to (30) by a factor of e in 0.22 orbit, which is close to the optimum. The optimum in the case of pitch motion was found by Zajac[5] to be 0.137 orbit. Equations of motion (20)† depend only on the above dimensionless parameters and are independent of I_1 and Ω as long as t is measured in fractions of an orbital period. Some initial conditions which might simulate a micrometeoroid impact or the

† Equations (20) were programmed on an IBM 7090 by Mrs. W. L. Mammel.

motion after the erection of the rods are at $t = 0$: $\xi = \eta = \zeta = \alpha = \beta = 0$, $\chi = 1$, $\omega_1 = \Omega$, $\omega_2 = 5\Omega$, $\omega_5 = \Omega$, $\omega_3 = \omega_4 = \omega_6 = 0$. Figs. 5–8 represent the computer solution of equations (20) using a magnetic hysteresis damper. In Fig. 5, θ is the angle between the z_1-axis and the local vertical. The satellite stops tumbling after four orbits and settles to within 10° of the local vertical after six orbits. The satellite librates about a cocked equilibrium position indefinitely due to the forcing torque of orbital eccentricity ($\epsilon = 0.01$). The pitch angular speed of body 1, ω_2, approaches one revolution per orbit, which is the proper speed for an earth-pointing satellite. Figs. 9–12 show similar results of a viscous damper. In this case the satellite ended up in an inverted position.

Effects of the environmental disturbing torques, such as those due to solar radiation and the interaction of the magnetic moment in the satellite with the geomagnetic field, have been investigated, although the results are not included here. Cases with various other initial conditions

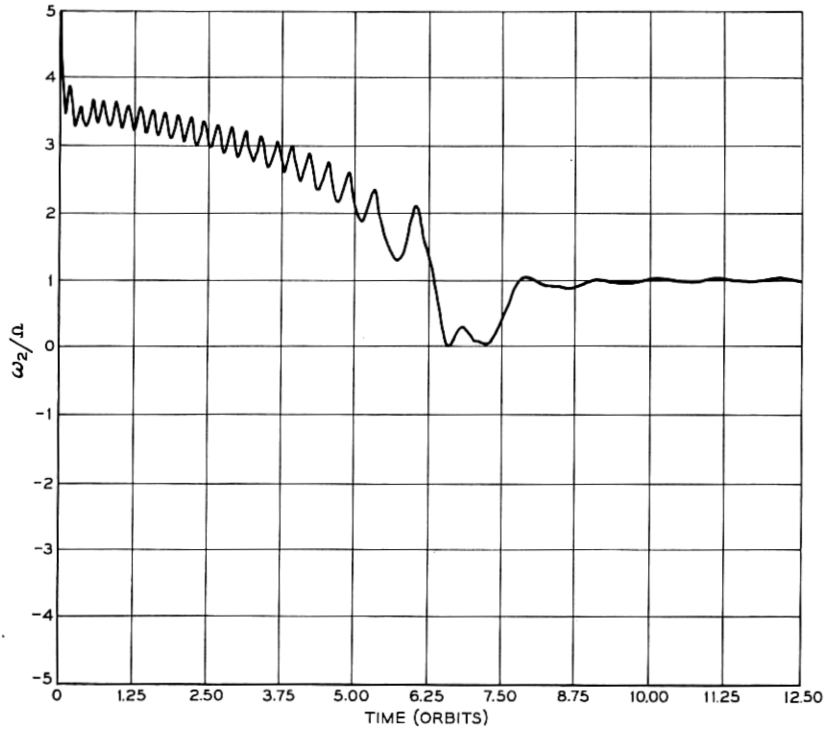

Fig. 12 — Component of angular velocity of the satellite along the y_1-axis for a viscous damper, ω_2/Ω.

have also been computed. All these results indicate that gravitational orientation of a two-body satellite is feasible.

APPENDIX

Nomenclature

A.1 *Latin Symbols*

a_{ij} = direction cosines of S_1-$x_1y_1z_1$ frame with respect to S_0-xyz frame (i,j = 1,2,3)
b_{ij} = direction cosines of S_2-$x_2y_2z_2$ frame with respect to S_1-$x_1y_1z_1$ frame (i,j = 1,2,3)
b_i = coefficients of characteristic equation of ξ_1, ξ_2, ζ (i = 0, 1, $\cdots$, 6)
B_i = complex constant of η_i (i = 1,2)
C = cosine operator
C_i = viscous damping constants of α,β (i = 1,2)
C_i', C_i'' = adjusted damping constants of α,β defined in equations (29) (i = 1,2)
d_i = coefficients defined in equations (29) (i = 1,2)
f_i = moment of inertia coefficients defined in equations (29) (i = 1,2)
$\mathbf{F}_H$ = force on body 1 due to reaction of hinge
$\mathbf{F}_i$ = resultant force on body i exclusive of $\mathbf{F}_H$ (i = 1,2)
$\mathbf{F}_i'$ = resultant force on body i exclusive of gravity and $\mathbf{F}_H$ (i = 1,2)
g = acceleration of gravity on the earth's surface
G = quantity defined in equation (19)
$\mathbf{G}_i$ = gravitational force on body i (i = 1,2)
H = hinge point
$\mathbf{I}$ = unit dyadic
I_i = adjusted moments of inertia (i = 1, $\cdots$, 6)
I_x, I_y, I_z = moments of inertia of composite body about the common center of mass
J_m = angular momentum of inertia wheel
k_i = spring constants producing torques in x_1, y_2 directions (i = 1,2)
k_i',$\bar{k}_i$ = adjusted spring constants defined in equations (29) (i = 1,2)
l = maximum linear dimension of the satellite

$\mathfrak{L}_i$ = position vector of center of mass of body i from hinge ($i = 1,2$)
ℓ_i = magnitude of $\mathfrak{L}_i$ ($i = 1,2$)
L_i = coefficients defined in equations (29) ($i = 1,2$)
m_i = mass of body i ($i = 1,2$)
m = total mass of satellite
$\bar{m}$ = reduced mass
n_i = direction cosines of z-axis on S_1-$x_1y_1z_1$ and S_2-$x_2y_2z_2$ frames ($i = 1, \cdots, 6$)
N_i = coefficients defined in equations (29) ($i = 1,2$)
O = center of the earth
P_i = arbitrary point in body i ($i = 1,2$)
P_x, P_y, P_z = ratio of moments of inertia in equations (35)
q_i = coefficients defined in equations (29) ($i = 1,2$)
$\mathbf{r}_i$ = position vector of P_i from center of mass of body i ($i = 1,2$)
$\mathbf{R}_i$ = position vector of P_i from O ($i = 1,2$)
R_E = mean radius of the earth
S = sine operator
s = variable in characteristic equations
S_0 = center of mass of satellite
S_i = center of mass of body i ($i = 1,2$)
t = time variable
$\mathbf{T}_H$ = reaction torque transmitted through the joint on body 1
$\mathbf{T}_i$ = resultant torque on body i exclusive of $\mathbf{T}_H$ ($i = 1,2$)
$\mathbf{T}_i'$ = resultant torque on body i exclusive of $\mathbf{T}_H$ and gravitational torque ($i = 1,2$)
$\mathbf{T}_{Gi}$ = gravitational torque on body i ($i = 1,2$)
$\mathbf{T}_c$ = constraint torque of joint on body 1
$\mathbf{T}_d$ = dissipative torque of joint on body 1
$\bar{\mathbf{T}}_{di}$ = magnitude of saturated hysteresis torque of magnet i ($i = 1,2$)
T_{di}^* = value of T_{di} when $\dot{\alpha}$ ($i = 1$) and $\dot{\beta}$ ($i = 2$) last changed sign
$\mathbf{T}_r$ = elastic restoring torque of joint on body 1
u_i = coefficients defined in equations (29) ($i = 1,2$)
X,Y,Z = fixed frame coordinates
x,y,z = rotating frame coordinates
x_1, y_1, z_1 = body 1 coordinates
x_2, y_2, z_2 = body 2 coordinates.

A.2 *Greek Symbols*

α = relative angle of rotation of body 2 about x_1-axis
$\bar{\alpha}$ = constant of magnet 1
α^* = values of α when $\dot{\alpha}$ last changed sign
β = relative angle of rotation of body 2 about y_2-axis
$\bar{\beta}$ = constant of magnet 2
β^* = values of β when $\dot{\beta}$ last changed sign
δ = angle between x_1-axis and the inertia wheel axis
ϵ = eccentricity of the orbit
ζ = Euler parameter
ζ_i = infinitesimal angle about z_i-axis ($i = 1,2$)
η = Euler parameter
η_i = infinitesimal angle about y_i-axis ($i = 1,2$)
θ = angle between z_1-axis and the local vertical or z-axis
λ_i = components of the relative angular velocity of body 1 with respect to rotating frame ($i = 1,2,3$)
μ = a gravitational constant of the earth
ξ = Euler parameter
ξ_i = infinitesimal angle about x_i-axis ($i = 1,2$)
$\boldsymbol{\varrho}$ = position vector of S_0 from O
$\boldsymbol{\varrho}_i$ = position vector of S_i from O ($i = 1,2$)
$\boldsymbol{\Phi}_i$ = moment of inertia dyadic of body i ($i = 1,2$)
$\boldsymbol{\Phi}_i'$ = quasi moment of inertia dyadic of body i ($i = 1,2$)
χ = Euler parameter
ψ = true anomaly of ellipse
Ω = mean orbital angular speed of satellite
$\boldsymbol{\omega}_{\mathrm{I}}$, $\boldsymbol{\omega}_{\mathrm{II}}$ = angular velocity of body 1,2
ω_1, ω_2, ω_3 = components of $\boldsymbol{\omega}_{\mathrm{I}}$ along x_1, y_1, z_1 axes
ω_4, ω_5, ω_6 = components of $\boldsymbol{\omega}_{\mathrm{II}}$ along x_2, y_2, z_2 axes
ω_r = natural frequency of an undamped roll libration.

A.3 *Notes*

$\hat{}$ = unit vector

$\cdot$ = time derivative in an inertial frame $\left(= \frac{d}{dt}\right)$

boldface characters indicate tensors and vectors (it is assumed that dropping the boldface means the magnitude of the vector; i.e., $\rho = |\boldsymbol{\varrho}|$).

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