Single Server Systems — II. Busy Periods

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This is the second of two papers dealing with single server systems. Statistical problems associated with the busy periods, i.e., the periods during which the server is continuously busy, are considered in the present paper. The input and the service time distributions may be quite general, and the length of the waiting line is unrestricted. Among the new results is an asymptotic expression for the probability density of the lengths of the busy periods. This expression holds when the arrival rate is almost equal to the service rate and when the busy periods tend to be long. It is hoped that the methods used here will throw additional light on known results.

I. INTRODUCTION

This is the second of two papers dealing with single server systems. In the first paper,[1] two subjects were discussed, (i) a method which led to the average values of several quantities of interest and (ii) the statistical behavior of a single server loss system. The present paper is concerned with the busy periods (the periods during which the server is continuously busy) in a single server system when no restrictions are placed on the queue length.

The distribution of the busy period lengths has been studied by a number of investigators, among them E. Borel,[2] D. G. Kendall,[3] F. Pollaczek,[4,5] L. Takács,[6,7] and B. W. Connolly.[8,9]

A closely related problem is that of the storage of water behind a dam. An interesting survey of this subject has been given by J. Gani.[10] The moving server problem treated by McMillan and Riordan[11] and by Karlin, Miller and Prabhu[12] is also related to the busy period problem. Again, for Poisson input, the distribution of the busy period lengths is related to the distribution of the delays in "last come, first served" type of service (see Riordan[20] and the references to earlier work given there).

The most general results are those due to Pollaczek. His work leads to the joint distribution function of n,S,y where n is the number of services comprising the busy period, S the length of the period, and y the length of the following idle period.

The object of the present paper is to obtain, in another way, results equivalent to those derived by Pollaczek. Integral equations similar to Lindley's[13] equation for the waiting time distribution are first set up. These equations are then solved by methods similar to the Wiener-Hopf technique used by W. L. Smith[14] to solve Lindley's equation. The aim of the presentation is to illustrate the methods; the discussion is heuristic; and no claims are made for completeness or rigor. On the other hand, it is hoped that the different point of view will add something to the understanding of the earlier results.

In Section II a brief review of some of the earlier results is given. In Section III the length of the busy period, as measured by the number n of customers served, is considered. The results of Section III are generalized in Section IV to apply to the joint distribution of n, S, and y. Special attention is paid to the distribution of S. The transient behavior of the queue length is considered in Section V. Since the method used here to study the busy times is closely related to Smith's method of dealing with waiting times, a review of the waiting time problem is given in the Appendix. Some changes are made in Smith's method in order to make it fit in better with the busy problem.

Throughout the paper, except possibly for (2), the intervals between arrivals are considered to be independent of each other and of the service times (recurrent input). The service times are also taken to be independent of each other and of the input.

As mentioned in the companion paper, I am indebted to John Riordan for much help in the preparation of this paper. I am also indebted to V. Beneš and L. Takács for helpful discussions and a number of references.

II. PRELIMINARY REMARKS

In this paper, $A(t)$ will denote the distribution function for the intervals between successive arrivals, $B(t)$ the service time distribution, and $A'(t)$,$B'(t)$ the respective probability densities. The arrival rate is a, the service rate is b, the average interval between arrivals is

$$a^{-1} = \int_0^\infty [1 - A(t)]\, dt$$

and the average service time is

$$b^{-1} = \int_0^\infty [1 - B(t)]\, dt.$$

The Laplace-Stieltjes transforms defined by

$$\alpha(s) = \int_0^\infty e^{-st}\, dA(t)$$
$$\beta(s) = \int_0^\infty e^{-st}\, dB(t) \tag{1}$$

play an important role.

Some information on the average length of a busy period (chosen at random from the universe of busy periods) may be obtained from the general results of the companion paper.[1] Thus, from Table I of that paper, when the service rate b exceeds the arrival rate a so that statistical equilibrium exists, the average length $\bar{S}$ of a busy period and the average number $\bar{n}$ of customers served (during a busy period) are given by

$$\bar{S} = \frac{a\tau}{b - a} = \frac{1}{bW(0)}, \qquad \bar{n} = \frac{1}{W(0)}. \tag{2}$$

Here the system capacity N is infinite, τ is the average length of an idle period, and $W(0)$ is the chance that a customer is served immediately upon arrival (zero waiting time). For Poisson input τ is $1/a$ and

$$\bar{S} = \frac{1}{b - a}, \qquad \bar{n} = \frac{1}{1 - \rho}, \qquad \rho = \frac{a}{b} \tag{3}$$

irrespective of the service time distribution. For inputs other than Poisson, one must know either τ or $W(0)$ in order to compute $\bar{S}$ from (2).

For the sake of orientation we state the following known results.

i. For Poisson input, general service and $a < b$, Kendall[3] has shown that the transform

$$\gamma(z) = \int_0^\infty e^{-zS}\, dG(S) \tag{4}$$

of the distribution function $G(S)$ of the busy period lengths satisfies the functional equation

$$\gamma(z) = \beta[z + a - a\gamma(z)]. \tag{5}$$

Here $\beta(s)$ is defined by (1). For Poisson input and exponential service, this leads to the probability density

$$G'(S) = \frac{dG(S)}{dS} = b(S\sqrt{ab})^{-1} I_1(2S\sqrt{ab})\, e^{-(b+a)S} \tag{6}$$

where $I_1(z)$ denotes a Bessel function of the first kind for imaginary argument.

ii. Suppose that we are given a busy period which has just begun. Let f_n be the chance that it will end after exactly n services. For Poisson input, Takács[6] has shown that the generating function

$$f(x) = \sum_1^\infty x^n f_n \tag{7}$$

must satisfy the functional equation

$$f(x) = x\beta[a - af(x)]. \tag{8}$$

For exponential service this leads to

$$f_n = \frac{(2n-2)!}{n!\,(n-1)!}\frac{\rho^{n-1}}{(1+\rho)^{2n-1}} \tag{9}$$

and for constant service time (Borel[2]) to

$$f_n = (\rho n)^{n-1}\frac{e^{-n\rho}}{n!}. \tag{10}$$

When $a < b$, all busy periods end eventually and $\sum_1^\infty f_n = 1$. When $a > b$, the customers arrive faster than the server can handle them, and sooner or later a busy period will start and never end. Given a busy period which has just begun, the probability that it will never end is $1 - \sum_1^\infty f_n$, where now $\sum_1^\infty f_n$ is less than 1. For Poisson input, exponential service, and $a > b$, the probability that the busy period will not end is $1 - (b/a)$.

iii. As mentioned in the Introduction, the results most closely related to the work of the present paper are those due to Pollaczek.[4,5] In particular he has shown the following. Given a busy period which has just begun, let $G_n'(y,S)dy\,dS$ be the chance that it will consist of exactly n services, have a length between S and $S + dS$, and be followed by an idle period whose length lies between y and $y + dy$. Let

$$\gamma_n(s,z) = \int_0^\infty dS \int_0^\infty dy\, e^{sy-zS} G_n'(y,S) \tag{11}$$

where e^{sy} appears instead of e^{-sy} for later convenience. Then, for rather general input and service distributions,

$$\sum_{n=1}^\infty x^n \gamma_n(s,z) = 1 - \exp\left\{\frac{1}{2\pi i}\int_C \frac{\ln[1 - x\beta(z+\zeta)\alpha(-\zeta)]\,d\zeta}{\zeta - s}\right\} \tag{12}$$

where $0 \leqq x \leqq 1$, $\mathrm{Re}(s) < \mathrm{Re}(\zeta) < 0 < \mathrm{Re}(z)$, $\mathrm{Re}(z + \zeta) > 0$, and the path of integration for ζ runs from $-i\infty$ to $+i\infty$ in the strip specified by the foregoing inequalities. In writing (11) and (12) it has been convenient to change from Pollaczek's notation to a notation resembling that used by Smith.[14]

III. NUMBER SERVED IN A BUSY PERIOD

3.1 *Derivation of Integral Equation*

Consider the busy period to start with the arrival of customer numnumber 1 at time 0, and let the service time of the rth arrival be s_r and t_r the interval between the arrivals of customers r and $r + 1$. The busy period consists of one service if $s_1 < t_1$, i.e., if $u_1 = s_1 - t_1 < 0$. It consists of two services if $s_1 \geqq t_1$ and $s_1 + s_2 < t_1 + t_2$, i.e., if $u_1 \geqq 0$ and $u_1 + u_2 < 0$. In general, n demands (customers) are served in a busy period if $U_1, U_2, \cdots, U_{n-1} \geqq 0$ and $U_n < 0$ where

$$\begin{aligned} U_n &= u_1 + u_2 + \cdots + u_n \\ u_r &= s_r - t_r . \end{aligned} \tag{13}$$

Since the u_r's are independent random variables whose distributions are known (indeed, ave $\exp(-su_r) = \beta(s)\alpha(-s)$) one may, in principle, find the joint distribution of $U_1, U_2, \cdots, U_n$. Then the probability f_n that n demands are served may be obtained by integration over the proper region in $U_1, \cdots, U_n$ space. However, it is more convenient to use an indirect method which depends upon the solution of an integral equation similar to that for the waiting time distribution.

Let $p_1(V)dV$ be the probability that $V < U_1 < V + dV$, and $p_n(V)dV$ the probability that $U_1, \cdots, U_{n-1} \geqq 0$ and $V < U_n < V + dV$, where V may be either positive or negative. Then

$$\begin{aligned} p_1(V) &= C'(V) \\ p_{n+1}(V) &= \int_0^\infty p_n(v)C'(V - v)\,dv \end{aligned} \tag{14}$$

where $C'(t) = dC(t)/dt$ is the probability density of $u_r = s_r - t_r$. Equations (14) lead to

$$J(x,V) = xC'(V) + x\int_0^\infty J(x,v)C'(V - v)\,dv \tag{15}$$

$$J(x,V) = \sum_{n=1}^{\infty} x^n p_n(V) \tag{16}$$

where the series and the integral are assumed to converge for $0 \leqq x \leqq 1$. Equation (15) is the integral equation which must be solved to obtain $J(x,V)$.

The integral of $p_n(V)$ from $V = -\infty$ to 0 gives the probability f_n that n demands are served and hence

$$f(x) = \sum_{n=1}^{\infty} x^n f_n = \int_{-\infty}^{0} J(x,V)\, dV. \tag{17}$$

When $a < b$, we expect all busy periods to end. Therefore

$$\sum_{1}^{\infty} f_n = 1 = \int_{-\infty}^{0} J(1,V)\, dV. \tag{18}$$

On the other hand, when the arrival rate a exceeds the service rate b we expect the queue length to increase with time, on the average. At the beginning of operations there may be a few busy periods, but eventually a busy period starts which does not end. This state of affairs is indicated by the fact that $\sum_1^\infty f_n$ is less than 1 when $a > b$. Thus, given a busy period which is just beginning, the probability that it will never end is

$$1 - \sum_{1}^{\infty} f_n = 1 - \int_{-\infty}^{0} J(1,V)\, dV. \tag{19}$$

The length of the idle period following a busy period of length n (services) is $-U_n$. The chance that a busy period will be of length n and will be followed by an idle period whose length lies between y and $y + dy$ is $p_n(-y)dy$. This is true for all values of a/b. When $a < b$, the chance that the length of an idle period picked at random from the universe of idle periods lies between y and $y + dy$ is

$$\sum_{1}^{\infty} p_n(-y)dy = J(1,-y)dy. \tag{20}$$

3.2 *Solution of Integral Equation*

The procedure used in the Appendix to solve Lindley's integral equation (95) suggests writing $J(x,V)$ as the sum of $J_-(x,V)$ and $J_+(x,V)$ where J_- is 0 for $V \geqq 0$ and J_+ is 0 for $V < 0$. Multiplying (15) by $\exp(-sV)$ and integrating with respect to V from $-\infty$ to $+\infty$ gives

$$\Phi_-(x,s) + \Phi_+(x,s) = x\beta(s)\alpha(-s)[1 + \Phi_+(x,s)] \tag{21}$$

where $\text{Re}(s)$ is confined to some suitable range and Φ_+ is the Laplace transform of J_+ and Φ_- is similarly related to J_- [cf. (97)].

Assume functions $\Psi_+(x,s)$, $\Psi_-(x,s)$ and positive numbers D_1, D_2 (which may depend on x) may be found such that

$$x\beta(s)\alpha(-s) - 1 = \frac{\Psi_+(x,s)}{\Psi_-(x,s)} \tag{22}$$

where

(*i*) $\Psi_+(x,s)$ is analytic in s and free from zeros in $\mathrm{Re}(s) > D_1 \geqq 0$,

(*ii*) $\Psi_-(x,s)$ is analytic and free from zeros in $\mathrm{Re}(s) < D_2$ with $D_2 > D_1$ when $0 < x \leqq 1$ and

(*iii*)

$$\Psi_+(x,s) \to s \quad \text{as} \quad |s| \to \infty \quad \text{in} \quad \mathrm{Re}(s) > D_1$$

$$\Psi_-(x,s) \to -s \quad \text{as} \quad |s| \to \infty \quad \text{in} \quad \mathrm{Re}(s) < D_2.$$

It is seen that $\Psi_+(x,s)$ and $\Psi_-(x,s)$ reduce to $\psi_+(s)$ and $\psi_-(s)$ of the Appendix when $x = 1$.

When $x\beta(s)\alpha(-s)$ is eliminated from (21) and (22) one obtains

$$\Phi_-\Psi_- - \Psi_- - s = \Phi_+\Psi_+ + \Psi_+ - s \tag{23}$$

where s has been subtracted from both sides in order to keep them finite at infinity. Considerations of analytic continuation and the analogy with (101) suggest that both sides are equal to some quantity $K(x)$ independent of s. When this turns out to be the case

$$\Phi_+(x,s) = \frac{[K(x) - \Psi_+(x,s) + s]}{\Psi_+(x,s)}.$$

Suppose for the moment that $b > a$. To determine $K(x)$ note (1) that $\Phi_+(x,s)$ is analytic in $\mathrm{Re}(s) \geqq 0$, and (2) that $\Psi_+(x,s)$ has a zero at $s = s_0 \equiv s_0(x)$ where $s_0 = 0$ when $x = 1$ and $s_0 \approx (1 - x)/(a^{-1} - b^{-1})$ when x is near 1. Also note that s_0 is positive when $b > a$ and x is slightly less than 1. Statement (1) is true since $\Phi_+(x,s)$ is the Laplace transform of $J_+(x,V)$ and the integral

$$\int_0^\infty J_+(x,V)\,dV = x\,\Pr(U_1 \geqq 0) + x^2 \Pr(U_1, U_2 \geqq 0) + \cdots < x(1 - x)^{-1}$$

converges. Statement (2) follows from relation (22) and the fact that the expansion of $x\beta(s)\alpha(-s) - 1$ in powers of s shows that it is zero at $s = s_0$. Hence either $\Psi_+(x,s_0)$ is zero or $\Psi_-(x,s_0)$ is infinite. The second possibility is ruled out if $1 - x$ is small enough to make $s_0 < D_2$ since $\Psi_-(x,s)$ is analytic in $\mathrm{Re}(s) < D_2$. Thus $K(x)$ must be taken to be

$-s_0$ in order to keep $\Phi_+(x,s)$ from having a pole in the region $\mathrm{Re}(s) \geqq 0$ at $s = s_0$.

Equating the left-hand side of (23) to $-s_0$ and solving for Φ_- gives

$$\int_{-\infty}^{0} e^{-sV} J_-(x,V)\, dV = \Phi_-(x,s) = 1 + \frac{s - s_0}{\Psi_-(x,s)} \tag{24}$$

$$J_-(x,V) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{sV} \Phi_-(x,s)\, ds \tag{25}$$

where c is chosen so that the path of integration in the s-plane passes to the left of the singularities of $\Phi_-(x,s)$. Comparison of (24) and (17) leads to

$$\sum_{n=1} x^n f_n = \Phi_-(x,0) = 1 - \frac{s_0}{\Psi_-(x,0)} = 1 + \frac{(1 - x)s_0}{\Psi_+(x,0)}. \tag{26}$$

In this result, $\Psi_-(x,s)$ and $\Psi_+(x,s)$ are obtained from (22); and s_0, a function of x, is that zero of $x\beta(s)\alpha(-s) - 1$ which approaches $s = 0$ as $x \to 1$. In the foregoing definition of s_0, b is supposed to exceed a. When a and b are in any ratio, the statement is amended to read, "$s = s_0$ is the one and only zero of $\Psi_+(x,s)$ in $\mathrm{Re}\, s > 0$ when $0 \leqq x < 1$." When $a \leqq b$, s_0 tends to 0 as $x \to 1$, and when $a > b$, s_0 tends to a positive number as $x \to 1$. These statements about s_0 are made on the strength of the examples given below in Section 3.3 and hence cannot be regarded as proved in general.

When $a < b$, (26) gives $\Phi_-(1,0) = 1$ in agreement with (18). When $a > b$, s_0 is not zero for $x = 1$, $\Phi_-(1,0)$ is less than 1, and (19) shows that the probability that a busy period will not end is

$$1 - \Phi_-(1,0) = \frac{s_0}{\Psi_-(1,0)} \qquad (a > b) \tag{27}$$

The relations corresponding to (24) and (25) for $J_+(v,V)$ are

$$\int_{0}^{\infty} e^{-sV} J_+(x,V)\, dV = \Phi_+(x,s) = -1 + \frac{s - s_0}{\Psi_+(x,s)} \tag{28}$$

$$J_+(x,V) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{sV} \Phi_+(x,s)\, ds \tag{29}$$

where c is chosen so that the path of integration passes to the right of the singularities of $\Phi_+(x,s)$.

3.3 *Examples*

The special cases used as examples in the Appendix to illustrate the waiting time distribution will be used here for the busy period length. It has been pointed out earlier that $\Psi_{\pm}(x,s)$ reduces to $\psi_{\pm}(s)$ for $x = 1$.

Example a. Poisson Input, Exponential Service. Equation (22) to determine the Ψ's becomes

$$\begin{aligned}\frac{\Psi_+(x,s)}{\Psi_-(x,s)} &= \frac{xab - (a - s)(b + s)}{(a - s)(b + s)} \\ &= \frac{s^2 + (b - a)s - (1 - x)ab}{(a - s)(b + s)} = \frac{(s - s_0)(s - s_1)}{(a - s)(b + s)}\end{aligned} \tag{30}$$

where

$$\left.\begin{matrix} s_0 \\ s_1 \end{matrix}\right\} = \frac{a - b}{2} \pm \tfrac{1}{2}\sqrt{(b + a)^2 - 4abx}. \tag{31}$$

As x runs from 0 to 1 the roots s_0 and s_1 move along the real axis in the s-plane as shown in Fig. 1.

The conditions for (22) lead us to take, as in (109),

$$\Psi_+(x,s) = \frac{(s - s_0)(s - s_1)}{(s + b)}, \qquad \Psi_-(x,s) = a - s \tag{32}$$

with $D_1 = s_0$ and $D_2 = a$. Expressions (26) give the generating function for the probability f_n that, given a busy period which has just begun, it will consist of exactly n services. Setting $s = 0$ in (32) to obtain $\Psi_-(x,0)$ leads to

$$\begin{aligned}\sum_{n=1} x^n f_n &= 1 - \frac{s_0}{\Psi_-(x,0)} = 1 - \frac{s_0}{a} \\ &= \frac{a + b}{2a} - \frac{1}{2a}\sqrt{(b + a)^2 - 4abx}.\end{aligned} \tag{33}$$

The coefficient of x^n in the power series expansion of the last expression

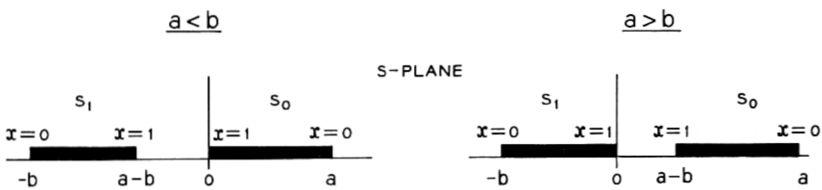

Fig. 1 — Ranges for s_0 and s_1 as x runs from 0 to 1.

gives the value (9) for f_n. When $x = 1$, the sum $\sum_1^\infty f_n$ is equal to 1 when $a < b$ and to b/a when $a > b$. Thus, from (19) or (27) the probability that the busy period will not end is $1 - (b/a)$, with $b < a$.

Inserting the expression

$$\Phi_-(x,s) = 1 + \frac{s - s_0}{a - s} = \frac{a - s_0}{a - s} \tag{34}$$

obtained from (24) in the integral (25), and evaluating the integral by closing the path of integration on the right when $V < 0$, shows that

$$J_-(x,V) = \begin{cases} (a - s_0)e^{aV} & (V < 0) \\ 0 & (V \geqq 0) \end{cases}. \tag{35}$$

From (20), the probability density for the lengths of the idle periods when $a < b$ is

$$J_-(1,-y) = a\, e^{-ay}$$

as expected for Poisson arrivals. When $a > b$, the idle period probability density is still given by the expression on the right, but now $J_-(1,-y)$ has to be divided by the normalizing factor $1 - (s_0/a)$ in which $x = 1$, i.e., by b/a, the probability that the busy period will end.

Although $\Phi_+(x,s)$, $J_+(x,V)$ are not needed to calculate the probabilities given above, their values as obtained from (28) and (29) will be stated for the sake of completeness

$$\Phi_+(x,s) = \frac{b + s_1}{s - s_1}$$

$$J_+(x,V) = \begin{cases} (b + s_1)e^{s_1 V} & (V \geqq 0) \\ 0 & (V < 0) \end{cases}.$$

Note that $b + s_1 = a - s_0$.

Example b. Poisson Input, General Service (Takács[6]). In this case, (22) and the related conditions are satisfied by the analogue of (110)

$$\Psi_+(x,s) = s - a + ax\beta(s), \qquad \Psi_-(x,s) = a - s \tag{36}$$

with $D_1 = s_0$ and $D_2 = a$. With the help of Rouche's theorem it may be shown that $\Psi_+(x,s)$ has only one zero, $s = s_0$, in Re $s > 0$ when $0 \leqq x < 1$. Furthermore, since $\Psi_+(x,0)$ is negative and $\Psi_+(x,a)$ is nonnegative, $0 < s_0 \leqq a$. Equations (24) and (25) show that $\Phi_-(x,s)$ and $J_-(x,V)$ are given by (34) and (35), just as for Poisson input and exponential service, and we still have

$$\sum x^n f_n = 1 - \frac{s_0}{a}. \tag{37}$$

However, s_0 no longer has the simple form (31), but it still tends to 0 if $a \leqq b$ and to a positive number if $a > b$ as $x \to 1$.

When Lagrange's expansion theorem is applied to $1 - (s_0/a)$ it is found that (see Pollaczek,[5] p. 102, Eq. (8.37))

$$f_n = \frac{(-a)^{n-1}}{n!} \left[\frac{d^{n-1}}{ds^{n-1}} \{\beta(s)\}^n \right]_{s=a} \tag{38}$$

which gives (9) and (10) as special cases. It should be recalled that from (3) the average number served in a busy period, for Poisson input and $a > b$, is $\bar{n} = 1/(1 - \rho)$.

Example c. Recurrent Input, Exponential Service. When one takes the steps leading to (111) (which pertains to the waiting time distribution for this case) as a guide, he is led to

$$\Psi_+(x,s) = \frac{(s - s_0)(s - s_1)}{b + s}, \qquad \Psi_-(x,s) = \frac{(s - s_0)(s - s_1)}{xb\alpha(-s) - b - s} \tag{39}$$

where s_1 is the only zero of

$$h(s) = s + b - xb\alpha(-s) \qquad (0 < x < 1) \tag{40}$$

which lies in $\mathrm{Re}(s) > 0$ and s_0 is the left-most zero of $h(s)$ in $\mathrm{Re}(s) > 0$ when x is close to 1. The existence of s_1 may be established with the help of Rouche's theorem. Then $h(0) = (1 - x)b > 0$ and $h(-b) = -x\alpha(b) > 0$ show that $-b > s_1 > 0$. To make the existence of s_0 plausible, consider the case when a is nearly equal to b and x is close to 1. When s is small, the series (114) for $\alpha(-s)$ gives

$$h(s) = (1 - x)b + s(1 - xba^{-1}) - \frac{xba_2s^2}{2} + \cdots. \tag{41}$$

It is seen that $h(s)$ has a double zero at $s = 0$ when $x = 1$ and $a = b$. When $x = 1 - \epsilon$, with ϵ small and positive, and $1 - ba^{-1} = \eta$ is small, the double zero splits into two simple zeros given approximately by

$$0 = -\epsilon b - s\eta + \frac{ba_2s^2}{2}.$$

The two roots of this equation are small, real, and of opposite sign. The positive root corresponds to s_0 and the negative one to s_1.

From (26)

$$\sum_1^\infty x^n f_n = 1 + \frac{(1 - x)s_0}{\Psi_+(x,0)} = 1 + \frac{(1 - x)b}{s_1} \tag{42}$$

and Lagrange's expansion theorem gives (see Pollaczek,[5] p. 103, Eq. (8.42))

$$f_n = c_{n-1} - c_n \qquad (c_0 = 1)$$
$$c_n = \frac{b^{n+1}}{n!}\left[\frac{d^{n-1}}{ds^{n-1}}\{s^{-2}[\alpha(-s)]^n\}\right]_{s=-b}. \tag{43}$$

When $a < b$, the average number served in a busy period may be obtained either by differentiating (42) and setting $x = 1$, or using the value of $W(0)$ obtained just below (111). Both methods give

$$\bar{n} = \frac{1}{W(0)} = \left[-\frac{b}{s_1}\right]_{x=1}. \tag{44}$$

Example d. Erlangian Input, Erlangian Service. Let $f(s)$ denote the same polynomial in s as in Example d of the Appendix. Then

$$x\beta(s)\alpha(-s) - 1 = x\left(1 + \frac{s}{bk}\right)^{-k}\left(1 - \frac{s}{al}\right)^{-l} - 1 = \frac{x - f(s)}{f(s)}.$$

When $x = 0$ the polynomial $x - f(s)$ has a zero of order k at $s = -bk$ and a zero of order l at $s = al$. Now let $0 < x < 1$. From Rouche's theorem and the fact that $|f(s)| > x$ on the imaginary s axis, it may be shown that $x - f(s)$ has k zeros in $\mathrm{Re}(s) < 0$ and l zeros in $\mathrm{Re}(s) > 0$. Denote the zeros in $\mathrm{Re}(s) < 0$ by $s_1, \cdots, s_k$, the left-most zero (when x is close to 1) in $\mathrm{Re}(s) > 0$ by s_0, and the remaining zeros in $\mathrm{Re}(s) > 0$ by $s_{k+1}, \cdots, s_{k+l-1}$. Then (22) takes the form

$$\frac{x - f(s)}{f(s)} = -\frac{(s - s_0)(s - s_1)\cdots(s - s_{k+l-1})}{(s - al)^l(s + bk)^k} = \frac{\Psi_+(x,s)}{\Psi_-(x,s)} \tag{45}$$

and the conditions on the Ψ's are satisfied by

$$\Psi_+(x,s) = \frac{(s - s_0)(s - s_1)\cdots(s - s_k)}{(s + bk)^k}$$
$$\Psi_-(x,s) = -\frac{(s - al)^l}{(s - s_{k+1})\cdots(s - s_{k+l-1})}. \tag{46}$$

From (26) the generating function for the number served in a busy period is

$$\sum_1^\infty x^n f_n = 1 - (al)^{-l} s_0 s_{k+1} s_{k+2} \cdots s_{k+l-1} = 1 - \frac{(1 - x)(-bk)^k}{s_1 \cdots s_k}$$

where $s_0, s_1, \cdots, s_{k+l-1}$ are functions of x.

IV. DURATION OF BUSY PERIOD

The distribution of the busy period lengths for recurrent input and general service may be obtained by an extension of the foregoing analysis. A number of the steps will be omitted here because of the similarity with the work of Section III. For the sake of simplicity, throughout this section it will be assumed that $a < b$ and hence that all busy periods eventually end.

4.1 *Derivation of Integral Equation*

Let $p_n(V,S)dV\,dS$ be the probability that $U_1, \cdots, U_{n-1} \geqq 0$, $V < U_n < V + dV$ and $S < S_n < S + dS$ where U_n is defined by (13) and S_n is the sum $s_1 + s_2 + \cdots + s_n$ of the first n service times. It may be shown that $p_n(V,S)$ is zero for $V > S$ and

$$p_1(V,S) = B'(S)A'(S - V)$$

$$p_{n+1}(V,S) = \int_0^S d\sigma \int_0^\sigma dv p_n(v,\sigma)B'(S - \sigma)A'(S - V - \sigma + v).$$

These may be combined to give the integral equation

$$\begin{aligned} J(x,V,S) &= xB'(S)A'(S - V) \\ &\quad + x\int_0^S d\sigma \int_0^\sigma dvJ(x,v,\sigma)B'(S - \sigma)A'(S - V - \sigma + v) \end{aligned} \tag{47}$$

where

$$J(x,V,S) = \sum_{n=1}^{\infty} x^n p_n(V,S). \tag{48}$$

The probability that a busy period will consist of exactly n services, have a length between S and $S + dS$, and be followed by an idle period of length between y and $y + dy$ is

$$p_n(-y,S)dS\,dy = G_n'(y,S)dS\,dy \tag{49}$$

where $G_n'(y,S)$ is the density function introduced in Section II. The probability that a busy period will consist of exactly n services and have a length between S and $S + dS$ is dS times

$$G_n'(S) = \int_{-\infty}^0 p_n(V,S)\,dV. \tag{50}$$

Summing with respect to n shows that the probability of a busy period length between S and $S + dS$ is dS times

$$G'(S) = \int_{-\infty}^{0} J(1,V,S)\, dV. \tag{51}$$

Furthermore, the probability the busy period consists of exactly n services is

$$f_n = \int_0^\infty dS \int_{-\infty}^{0} p_n(V,S)\, dV.$$

4.2 *Solution of Integral Equations*

Multiplying (47) by $\exp[-zS - sV]$, and integrating V from $-\infty$ to S and S from 0 to ∞ gives

$$\begin{aligned}\Phi_-(x,s,z) + \Phi_+(x,s,z) &= x\beta(s + z)\alpha(-s)[1 + \Phi_+(x,s,z)]\\ \Phi_+(x,s,z) &= \int_0^\infty dS \int_0^S dV\, e^{-zS-sV} J(x,V,S)\\ \Phi_-(x,s,z) &= \int_0^\infty dS \int_{-\infty}^{0} dV\, e^{-zS-sV} J(x,V,S).\end{aligned} \tag{52}$$

From the last equation

$$\int_{-\infty}^{0} e^{-sV} J(x,V,S)\, dV = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{zS} \Phi_-(x,s,z)\, dz \tag{53}$$

where c is such that the singularities of the integrand lie to the left of the path of integration.

Assume the factorization

$$x\beta(s + z)\alpha(-s) - 1 = \frac{\Psi_+(x,s,z)}{\Psi_-(x,s,z)} \tag{54}$$

where the Ψ functions satisfy conditions (i), (ii), (iii) set forth in connection with (22), z being regarded as an imaginary constant (or is at least on the path of integration $\mathrm{Re}(z) = c$ in (53)). When $z = 0$, $\Psi_\pm(x,s,z)$ reduces to the $\Psi_\pm(x,s)$ of (22) and when $z = 0$ and $x = 1$, to the $\psi_\pm(s)$ of (99). The analogue of (24) turns out to be

$$\Phi_-(x,s,z) = 1 + \frac{s - s_0(x,z)}{\Psi_-(x,s,z)} \tag{55}$$

where $s = s_0(x,z)$ is that root of

$$x\beta(s + z)\alpha(-s) - 1 = 0 \tag{56}$$

which tends to $s = 0$ as $x \to 1$ and $z \to 0$.

The various probabilities of interest may be computed, at least in theory, from $\Phi_-(x,s,z)$. The relations are indicated in Table I. The Laplace transforms $\gamma_n(s,z)$, $\gamma_n(z)$, $\gamma(z)$ are the same as those introduced in Section II.

From (55) and the last two entries in the table it follows that the Laplace transforms of $\sum x^n G_n'(S)$ and $G'(S)$ are respectively

$$\Phi_-(x,0,z) = 1 - \frac{s_0(x,z)}{\Psi_-(x,0,z)} = 1 + \frac{[1 - x\beta(z)]s_0(x,z)}{\Psi_+(x,0,z)} \tag{57}$$

$$\gamma(z) = \Phi_-(1,0,z) = 1 - \frac{s_0(1,z)}{\Psi_-(1,0,z)} = 1 + \frac{[1 - \beta(z)]s_0(1,z)}{\Psi_+(1,0,z)}. \tag{58}$$

Furthermore, the generating function (26) for f_n may also be written as $\Phi_-(x,0,0)$.

4.3 *Examples*

The special cases used earlier will again serve as examples. The results of Examples *b* and *c* are equivalent to results given by Pollaczek.

TABLE I

Probability Function	Laplace Transform
$G_n'(y,S) = p_n(-y,S)$	$\gamma_n(s,z) = \text{Expression (11)} = \int_0^\infty dS \int_{-\infty}^0 dV\, e^{-zS-sV} p_n(V,S)$
$\sum_1^\infty x^n G_n'(y,S) = J(x,-y,S)$	$\sum_1^\infty x^n \gamma_n(s,z) = \Phi_-(x,s,z)$
$G_n'(S)$	$\int_0^\infty e^{-zS} G_n'(S)\, dS = \gamma_n(0,z) = \gamma_n(z)$
$\sum_1^\infty x^n G_n'(S) = \int_{-\infty}^0 J(x,V,S)\, dV$	$\sum_1^n x^n \gamma_n(z) = \Phi_-(x,0,z)$
$G'(S) = \int_{-\infty}^0 J(1,V,S)\, dV$	$\int_0^\infty e^{-zS} G'(S)\, dS = \gamma(z) = \Phi_-(1,0,z)$

Example a. Poisson Input, Exponential Service. In this case (54) becomes

$$\frac{\Psi_+(x,s,z)}{\Psi_-(x,s,z)} = \frac{s^2 + s(z + b - a) - a(z + b - xb)}{(s + b + z)(a - s)} = \frac{(s - s_0)(s - s_1)}{(s + b + z)(a - s)} \tag{59}$$

where $s_j \equiv s_j(x,z)$ is given by

$$\left.\begin{matrix} s_0 \\ s_1 \end{matrix}\right\} = \frac{a - b - z}{2} \pm \frac{1}{2}\sqrt{(b + a + z)^2 - 4xab}. \tag{60}$$

When x is fixed and z runs from $-i\infty$ to $+i\infty$, s_0 traverses an oval-shaped path in the s-plane. The oval lies in $\text{Re}(s) \geqq 0$, it starts and ends at $s = a$ (corresponding to $z = \pm i\infty$), and when $z = 0$ it crosses the real s-axis between $s = 0$ and $s = a$ as indicated by Fig. 1, case $a < b$. At the same time, s_1 traverses a path roughly parallel to the imaginary axis. The path lies in the region $\text{Re}(s) < 0$, is asymptotic to the line $\text{Re}(s) = -b$ at $z = \pm i\infty$, and crosses the real s-axis between $s = -b$ and $s = a - b$. As $x \to 0$, the oval shrinks to the point $s = a$ and the path of s_1 tends to the straight line $\text{Re}(s) = -b$.

This behavior of s_0 and s_1 and the similarity with (32) lead us to take

$$\Psi_+(x,s,z) = \frac{(s - s_0)(s - s_1)}{s + b + z}, \qquad \Psi_-(x,s,z) = a - s. \tag{61}$$

From (57)

$$\Phi_-(x,0,z) = 1 - \frac{s_0(x,z)}{a} = \frac{1}{2a}[b + a + z - \sqrt{(b + a + z)^2 - 4xab}]. \tag{62}$$

Inverting this Laplace transform (see, for instance, Pair 556.1, Campbell and Foster[16]) gives (Takács[6])

$$\sum_1^\infty x^n G_n'(S) = S^{-1}\sqrt{\frac{xb}{a}}\, e^{-(b+a)S} I_1(S\sqrt{4xab}) \tag{63}$$

where $I_1(z)$ is a Bessel function of the first kind for imaginary argument. Putting $x = 1$ in (63) gives Kendall's[3] expression (6) for $G'(S)$.

Example b. Poisson Input, General Service (Prabhu[17]). In analogy with (36)

$$\Psi_+(x,s,z) = s - a + xa\beta(s + z), \qquad \Psi_-(x,s,z) = a - s \tag{64}$$

and (57) gives

$$\Phi_-(x,0,z) = 1 - a^{-1}s_0(x,z) \tag{65}$$

where $s = s_0(x,z)$ is that root of

$$s - a + xa\beta(s + z) = 0 \tag{66}$$

which tends to zero as $x \to 1$ and $z \to 0$. Rouche's theorem may be used to show that if $0 \leqq x < 1$ and Re $z \geqq 0$, then (66) has one and only one root (namely, $s_0(x,z)$) in $\text{Re}(z + s) > 0$.

When s in (66) is replaced by $s_0(x,z) = a - a\Phi_-$ (from (65) with $\Phi_- \equiv \Phi_-(x,0,z)$) one obtains

$$\Phi_- = x\beta(z + a - a\Phi_-). \tag{67}$$

This is a functional equation satisfied by the generating function

$$\Phi_- = \sum x^n \gamma_n(z). \tag{68}$$

Setting $x = 1$ gives Kendall's equation (5)[3]

$$\gamma(z) = \beta(z + a - a\gamma(z))$$

for the Laplace transform of $G'(S)$.

Example c. Recurrent Input, Exponential Service (Connolly,[8] Takács[7]). The analogue of (39) is

$$\Psi_+(x,s,z) = \frac{(s - s_0)(s - s_1)}{s + b + z},$$
$$\Psi_-(x,s,z) = \frac{(s - s_0)(s - s_1)}{xb\alpha(-s) - s - b - z} \tag{69}$$

where $s = s_1$ is the only root of

$$xb\alpha(-s) - s - b - z = 0 \tag{70}$$

which has a negative real part when $\text{Re}(z) \geqq 0$ and $0 \leqq x \leqq 1$, and $s = s_0$ is the root which tends to $s = 0$ when $z \to 0$ and $x \to 1$.

Equations (57) and (58) give

$$\Phi_-(x,0,z) = 1 + \frac{z + (1 - x)b}{s_1(x,z)}$$
$$\gamma(z) = 1 + \frac{z}{s_1(1,z)} \tag{71}$$

which may be used in Table I. The coefficients of $-z$ and $z^2/2$ in the power series expansion of $\gamma(z)$ give the first and second moments of the distribution of the busy period length S. With the help of (70) it is found that

$$\bar{S} = -\frac{1}{s_{10}} \tag{72}$$

where $\delta(t)$ is a unit impulse. Since the right-hand side of (75) is a continuous function of S, it cannot hold in this case. However, formal calculation of (75) gives

$$G'(S) \approx S^{-3/2}(2\pi b)^{-\frac{1}{2}} e^{-bS(1-\rho)^2/2}. \tag{77}$$

By using $n! \sim \sqrt{2\pi n}\, n^n e^{-n}$ and $\rho \approx 1$ it may be shown that (77) becomes a smoothed version of (76) when S is replaced by nb^{-1}. A better approximation to (76) may be obtained by noting that the Laplace transform $\gamma(z)$ of (76) is of the form $\sum f_n \exp(-znb^{-1})$. Hence $\gamma(z)$ is a periodic function of z with period $i2\pi b$, and contributions to (73) may be expected not only from the region around $z = 0$ but also from regions near $z = i2\pi kb$ where $k = \cdots, -1, 0, 1, 2, \cdots$. It may be shown that the sum of these contributions does indeed approximate (76) when n is large and $p \approx 1$.

The approximation (75) holds for large values of S. One may obtain an idea of the behavior of $G'(S)$ for small values of S by noting that now the busy period contains only a small number of services. In particular, the assumption that the very short busy periods consist of a single service leads to the approximation $G'(S) \approx G_1'(S) = B'(S)[1 - A(S)]$. Higher-order approximations may be obtained by computing $p_1(V,S)$, $p_2(V,S)$, $\cdots$ step by step and using (50). The first step gives the result cited, namely

$$G_1'(S) = \int_{-\infty}^{0} B'(S)A'(S - V)\, dV = B'(S)[1 - A(S)]. \tag{78}$$

V. GROWTH OF QUEUE

In Section III the functions $\Psi_{\pm}(x,s)$ were introduced to obtain the chance f_n that a busy period will consist of n services. It is interesting to note that these functions may also be used to determine $W_r(t)$, the waiting time distribution for the rth customer. Only a sketch of the procedure is given here. More complete information on $W_r(t)$ is given by Pollaczek,[5] Spitzer,[18] and Lindley.[13]

so that $L_+(x,t) = 0$ for $t < 0$, it is found from (94) that

$$L_+(x,t) = W_0(t) + x \int_{-\infty}^{\infty} L_+(x,t-\tau)\, dC(\tau). \tag{80}$$

This holds only for $t \geqq 0$. To solve the integral equation, let $L_-(x,t)$ denote the value of the right-hand side for $t < 0$ and set $L_-(x,t) = 0$ for $t \geqq 0$. It is found that

$$\varphi_-(x,s) + \varphi_+(x,s) = s^{-1} + x\varphi_+(x,s)\beta(s)\alpha(-s) \tag{81}$$

where φ_+ is the Laplace transform of L_+ and φ_- is similarly related to L_- (cf. (97)). Equation (81) is similar to (21). When the functions $\Psi_\pm(x,s)$ appearing in (22) are introduced, the analogue of (23) turns out to be

$$\begin{aligned}\varphi_-(x,s)\Psi_-(x,s) - s^{-1}\Psi_-(x,s) + s^{-1}\Psi_-(x,0)\\ = \varphi_+(x,s)\Psi_+(x,s) + s^{-1}\Psi_-(x,0)\end{aligned} \tag{82}$$

where $s^{-1}\Psi_-(x,0)$ is added to both sides in order to cancel the pole of $s^{-1}\Psi_-(x,s)$ at $s = 0$. Setting both sides of (82) equal to a quantity $K(x)$ independent of s leads to

$$\varphi_+(x,s) = \frac{[s - s_0(x)]\Psi_-(x,0)}{s\Psi_+(x,s)s_0(x)}$$

where $s = s_0(x)$ is the one and only zero of $\Psi_+(x,s)$ in $\mathrm{Re}(s) > 0$.

Inversion gives the required result:

$$\sum_0^\infty x^r W_r(t) = \frac{\Psi_-(x,0)}{s_0(x)} \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{[s - s_0(x)]e^{st}}{s\Psi_+(x,s)}\, ds \qquad (\epsilon > 0). \tag{83}$$

5.2 *Poisson Input and Exponential Service*

For the special case of Poisson input and exponential service, the values of $\Psi_\pm(x,s)$ are given by (32). Insertion in (83) gives

$$\begin{aligned}\sum_0^\infty x^r W_r(t) &= \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{a(s+b)e^{st}}{s_0 s(s - s_1)}\, ds\\ &= \frac{1}{1-x} - \frac{(a+b)e^{-bt}}{2b(1-x)} (1 - \sqrt{1-cx})e^{\tau - \tau\sqrt{1-cx}}\end{aligned} \tag{84}$$

where $\epsilon > 0$, $t > 0$ and

$$c = \frac{4ab}{(a+b)^2}, \qquad \tau = \frac{t(b+a)}{2}. \tag{85}$$

Upon using

$$(1 - \sqrt{1 - cx})^{k+1} = \sum_{n=k}^{\infty} \frac{(cx)^{n+1}(2n-k)!(k+1)2^{k-1}}{(n+1)!(n-k)!4^n}$$
$$(1 - \sqrt{1 - cx})\exp[\tau - \tau\sqrt{1 - cx}] = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(cx)^{n+1}}{(n+1)!4^n} P(n,\tau) \tag{86}$$

it is found that

$$W_r(t) = 1 - \frac{ae^{-bt}}{a+b}\sum_{n=0}^{r-1} \frac{c^n P(n,\tau)}{(n+1)!4^n} \tag{87}$$

where $P(n,\tau)$ is the polynomial

$$P(n,\tau) = \sum_{k=0}^{n} \frac{(2n-k)!(k+1)(2\tau)^k}{k!(n-k)!}$$
$$= (2\tau)^{n+\frac{1}{2}} e^{\tau} \pi^{-\frac{1}{2}}[(1+\tau)K_{n+\frac{1}{2}}(\tau) - \tau K_{n-\frac{1}{2}}(\tau)] \tag{88}$$

and K denotes a Bessel function of the second kind for imaginary argument.[19] It may be shown that $P(n,\tau)$ is $0(4^n n!/\sqrt{n})$ as $n \to \infty$ with τ finite.

Setting $x = 1$ in (86) and using the result to transform (87) leads to

$$W_r(t) = 1 - \rho\, e^{-(b-a)t} + a\, e^{-bt}F_r(t) \qquad (a \leqq b)$$
$$W_r(t) = 0 + a\, e^{-bt}F_r(t) \qquad (a \geqq b)$$

where

$$F_r(t) = (a+b)^{-1} \sum_{n=r}^{\infty} \frac{c^n P(n,\tau)}{(n+1)!4^n}.$$

The value of $F_r(t)$ is unchanged when the values of a and b are interchanged.

When $a = b = 1$, differentiation of the generating function (84) leads to expressions for the probability density $W_r'(t)$, $r > 0$:

$$W_r'(t) = \frac{t^r}{2^r r!}\left(\frac{2t}{\pi}\right)^{\frac{1}{2}} [K_{r+\frac{1}{2}}(t) - K_{r-\frac{1}{2}}(t)]$$
$$= \frac{e^{-t}}{2^{2r-1}} \sum_{n=0}^{r-1} \frac{(2r-n-1)!(2t)^n}{r!n!(r-n-1)!} \tag{89}$$
$$= \frac{2}{\pi}\int_0^{\infty} \frac{\cos tu\, du}{(u^2+1)^r}\left[\frac{1}{u^2+1} - \frac{t}{2r}\right].$$

5.3 *Queue Behavior as $r \to \infty$*

Return now to the case of recurrent input and general service. When r becomes large $W_r(t)$ approaches the limit $W(t)$ discussed in the Appendix if $a < b$. Questions related to this approach have been studied by Lindley, Pollaczek, Spitzer and others.

When $a > b$, the customers arrive faster than the server can handle them and the waiting line tends to grow steadily. An idea of the behavior of $W_r(t)$ in this case may be obtained by using the notation of the Appendix. The rth customer arrives at time $T_r = t_0 + t_1 + \cdots + t_{r-1}$ and his service begins at $S_r + I_r$, where $S_r = s_0 + s_1 + \cdots + s_{r-1}$ and I_r is the total amount of time the server is idle in the interval $(0,T_r)$. The waiting time of the rth customer is $w_r = S_r + I_r - T_r$. Since the server may be expected to be continuously busy after a few initial idle periods, we expect I_r to approach some constant value as $r \to \infty$. Thus, I_r becomes small in comparison with $S_r - T_r = u_0 + u_1 + \cdots + u_{r-1}$. Upon making the approximation $w_r \approx S_r - T_r$ and using the central limit theorem, it is found that, for $a > b$ and $r \gg 1$

$$W_r'(t) = \frac{dW_r(t)}{dt} \approx \frac{1}{\sigma\sqrt{2\pi r}} \exp\left[-\frac{(t - \mu r)^2}{2r\sigma^2}\right]. \tag{90}$$

Here $\mu = b^{-1} - a^{-1}$ is the average value of $u_r = s_r - t_r$ and $\sigma^2 = v_B + v_A$ is its variance, v_B and v_A being the respective variances of s_r and t_r.

When $a = b$, the approximation (90) no longer holds. However, Pollaczek[5] has shown that, at least for Poisson input, in place of (90) we have

$$W_r'(t) \approx \frac{2}{\sigma\sqrt{2\pi r}} \exp\left[-\frac{t^2}{2r\sigma^2}\right] \qquad (t > 0). \tag{91}$$

This agrees with the asymptotic form of the integral in (89).

Appendix

*Waiting Time Distribution**

In Sections III and IV the busy period problem has been investigated by a method similar to the Wiener-Hopf technique used by Smith[14] to deal with the waiting time distribution. The application of the method to waiting time problems is reviewed in this Appendix. Smith's approach has been changed slightly in order to make it fit in better with the busy period problem.

* Here the "waiting time" of a customer is the interval between his arrival and the instant his service begins.

A.1 *Derivation of Integral Equation*

The basic integral equation is due to Lindley.[13] Suppose the single server system (with unrestricted queue length, recurrent input, and general service) starts operations when the 0th customer arrives at time 0. Following Lindley, write w_r for the waiting time of the rth customer, s_r for his service time, and t_r for the interval between the arrivals of customers r and $r + 1$. The rth customer stays in the system for an interval of length $w_r + s_r$, and the $(r + 1)$th customer arrives t_r units of time after this interval begins. If $w_r + s_r \leqq t_r$, the $(r + 1)$th customer does not have to wait and w_{r+1} is zero. When $w_r + s_r > t_r$, the $(r + 1)$th customer has to wait $w_r + s_r - t_r$ units of time. Thus

$$\begin{aligned} w_{r+1} &= 0 \qquad && (w_r + s_r \leqq t_r) \\ w_{r+1} &= w_r + s_r - t_r \qquad && (w_r + s_r > t_r). \end{aligned} \tag{92}$$

It is assumed that the s_r are independent random variables with the common distribution function $B(t)$, and also that the t_r are independent (of each other and of the s_r) with common distribution function $A(t)$. The rule for combining probability distributions shows that the distribution function $C(t)$ of the variable $u_r = s_r - t_r$ is given by

$$C(t) = \int_0^\infty B(t + \tau)\, dA(\tau). \tag{93}$$

Since w_r and u_r are independent random variables, it follows upon rewriting (92) as

$$\begin{aligned} w_{r+1} &= 0 \qquad && (w_r + u_r \leqq 0) \\ w_{r+1} &= w_r + u_r \qquad && (w_r + u_r > 0) \end{aligned}$$

that the distribution function for w_{r+1} is

$$W_{r+1}(t) = \int_{-\infty}^{t} W_r(t - \tau)\, dC(\tau) \qquad (t \geqq 0). \tag{94}$$

By starting with $W_0(t) = 1$ for $t \geqq 0$, which states that the 0th customer is served immediately, one may compute $W_1(t)$, $W_2(t)$, $\cdots$, in succession from (94). When the service rate b exceeds the arrival rate a, i.e., when $\rho = a/b < 1$, $W_r(t)$ tends to $W(t)$ as $r \to \infty$ where $W(t)$ satisfies Lindley's integral equation

$$W(t) = \int_{-\infty}^{t} W(t - \tau)\, dC(\tau) \qquad (t \geqq 0). \tag{95}$$

$W(t)$ is the distribution function for the waiting time when statistical equilibrium prevails. Note that (95) holds only for nonnegative t. When t is negative, $W(t)$ is zero but the integral on the right does not vanish.

A.2 *Solution of Integral Equation*

Let $W_-(t)$ be the value of the integral for $t < 0$ and take $W_-(t) = 0$ for $t \geqq 0$. Since $W(t - \tau)$ is zero for $\tau > t$, (95) may be written as

$$W_-(t) + W(t) = \int_{-\infty}^{\infty} W(t - \tau)C'(\tau)\, d\tau \tag{96}$$

which holds for all real values of t. The derivative $C'(\tau) = dC(\tau)/d\tau$ is the probability density of the random variable $u_r = s_r - t_r$.

Multiply both sides of (96) by $\exp(-st)$, where $0 < \mathrm{Re}(s) < D$ with D such that the following integrals converge. The existence of D is ensured by assumptions made below. Integrate from $t = -\infty$ to $t = \infty$ and introduce the transforms

$$\varphi_+(s) = \int_{-\infty}^{\infty} e^{-st}W(t)\, dt = \int_0^{\infty} e^{-st}W(t)\, dt$$

$$\varphi_-(s) = \int_{-\infty}^{\infty} e^{-st}W_-(t)\, dt = \int_{-\infty}^{0} e^{-st}W_-(t)\, dt \tag{97}$$

$$\int_{-\infty}^{\infty} e^{-st}C'(\tau)\, d\tau = \text{ave} \exp\,[-ss_r + st_r] = \beta(s)\alpha(-s)$$

where $\beta(s)$ and $\alpha(s)$ are the respective Laplace-Stieltjes transforms of the service and interarrival distribution functions $B(t)$ and $A(t)$. This carries (96) into

$$\begin{aligned} \varphi_-(s) + \varphi_+(s) &= \varphi_+(s)\beta(s)\alpha(-s) \\ \varphi_-(s) &= \varphi_+(s)[\beta(s)\alpha(-s) - 1]. \end{aligned} \tag{98}$$

Since $W(t)$ and $B(t)$ are distribution functions, both $\varphi_+(s)$ and $\beta(s)$ are analytic in the region $\mathrm{Re}(s) > 0$. To ensure convergence of the integrals in (97) involving $W_-(t)$ and $C'(\tau)$, assume that the probability density $A'(t) = dA(t)/dt$ exists and is $0[\exp(-Dt)]$ as $t \to \infty$, where D is positive but may be arbitrarily small. It may then be shown from (93) that $C(t)$ is $0[\exp(Dt)]$ as $t \to -\infty$ and, using this in (95), that $W_-(t)$ is also $0[\exp(Dt)]$ as $t \to -\infty$. It follows that both $\varphi_-(s)$ and $\alpha(-s) = \text{ave} \exp(st_r)$ are analytic in the region $\mathrm{Re}(s) < D$.

Now suppose that functions $\psi_+(s)$ and $\psi_-(s)$ may be found such that

$$\beta(s)\alpha(-s) - 1 = \frac{\psi_+(s)}{\psi_-(s)} \tag{99}$$

where (i) $\psi_+(s)$ is analytic and free from zeros in the half-plane $\text{Re}(s) > 0$, and (ii) $\psi_-(s)$ is analytic and free from zeros in $\text{Re}(s) < D$. Although these functions may be expressed as integrals when suitable conditions are satisfied (see, for example, Smith[14]) their expression in tractable form is usually the most difficult step in obtaining $W(t)$. For future convenience assume that $\psi_+(s)$ and $\psi_-(s)$ may be chosen so that

$$\begin{aligned} \psi_+(s) &\to s \quad \text{as} \quad |s| \to \infty \quad \text{in} \quad \text{Re}(s) > 0 \\ \psi_-(s) &\to -s \quad \text{as} \quad |s| \to \infty \quad \text{in} \quad \text{Re}(s) < D. \end{aligned} \tag{100}$$

The difference in sign is required by the fact that the left-hand side of (99) tends to -1 as $s \to \pm i\infty$ (unless both $A(t)$ and $B(t)$ have discontinuous jumps, a case we shall rule out).

When the resolution (99) is possible, (98) becomes

$$\varphi_-(s)\psi_-(s) = \varphi_+(s)\psi_+(s), \qquad 0 < \text{Re}(s) < D. \tag{101}$$

The right-hand side is analytic for $\text{Re}(s) > 0$ and the left hand for $\text{Re}(s) < D$. Equality in the strip implies that each is the analytic continuation of a function which has no singularities in the finite part of the s-plane. It turns out that when conditions (100) are satisfied, this function may be taken to be a constant K. Indeed, conditions (100) were imposed to make this so. Then the Laplace transform of $W(t)$ is

$$\varphi_+(s) = \frac{K}{\psi_+(s)} \tag{102}$$

which is analytic in $\text{Re}(s) > 0$ by virtue of the requirements on $\psi_+(s)$. Since

$$\lim_{s\to 0} s\varphi_+(s) = \lim_{s\to 0} \int_{-0}^{\infty} e^{-st}\, dW(t) = 1$$

it follows that $\psi_+(0)$ is 0 and K is given by

$$K = \lim_{s\to 0} \frac{\psi_+(s)}{s} = \left[\frac{d\psi_+(s)}{ds}\right]_{s=0} = \psi_+'(0). \tag{103}$$

The constant K is also equal to $W(0)$, the probability that a customer will not have to wait for service. One way to see this is to note that from (102) and (100)

$$\begin{aligned} K = \varphi_+(s)\psi_+(s) &= \lim_{s\to\infty} \varphi_+(s)s = \lim_{s\to\infty} s \int_0^{\infty} e^{-st} W(t)\, dt \\ &= \lim_{s\to\infty} \int_0^{\infty} e^{-u} W\left(\frac{u}{s}\right) du = W(0). \end{aligned} \tag{104}$$

The limit exists since $0 < W(t) \leqq 1$ and $W(t)$ decreases monotonically as t decreases to 0. Although $W(t)$ is discontinuous on the left at $t = 0$ and may have discontinuities for positive values of t (as it does when both $A(t)$ and $B(t)$ have discontinuities), we take it to be continuous on the right at $t = 0$.

Thus when $\psi_+(s)$ is known, $\varphi_+(s)$ is determined and $W(t)$ may be obtained by inversion,

$$W(t) = \frac{K}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{st}\, ds}{\psi_+(s)} \qquad (c > 0). \tag{105}$$

The Laplace-Stieltjes transform of $W(t)$ is

$$w(s) = \int_0^\infty e^{-st}\, dW(t) = s\varphi_+(s) = \frac{sK}{\psi_+(s)}. \tag{106}$$

As $s \to \infty$, $w(s)$ tends to $W(0) = K$. The value of $-dw(s)/ds$ at $s = 0$ is equal to the average waiting time.

In some cases $\psi_-(s)$ is simpler than $\psi_+(s)$ and in place of (102) one may use

$$\varphi_+(s) = \frac{K}{[\beta(s)\alpha(-s) - 1]\psi_-(s)} \tag{107}$$

where differentiation of (99) gives

$$K = \psi_+'(0) = (a^{-1} - b^{-1})\psi_-(0). \tag{108}$$

A.3 *Examples*

The results just obtained will be illustrated by several special cases, all of which have appeared in the literature.

Example a. Poisson Input, Exponential Service. Here $\alpha(s) = a/(a + s)$, $\beta(s) = b/(b + s)$, and $\psi_+(s)$, $\psi_-(s)$ are to be determined from (99), which now takes the form

$$\frac{ba}{(b + s)(a - s)} - 1 = \frac{s(s + b - a)}{(b + s)(a - s)} = \frac{\psi_+(s)}{\psi_-(s)}.$$

Inspection shows that when $D = a$

$$\psi_+(s) = \frac{s(s + b - a)}{b + s}, \qquad \psi_-(s) = a - s \tag{109}$$

satisfy the requirements set forth in connection with (99). Then (103),

(102) and (105) give

$$K = \lim_{s \to 0} \frac{\psi_+(s)}{s} = \frac{(b-a)}{b} = 1 - \rho$$

$$\varphi_+(s) = \frac{K}{\psi_+(s)} = \frac{(1-\rho)(b+s)}{s(b-a+s)} = \frac{1}{s} - \frac{\rho}{b-a+s}$$

$$W(t) = 1 - \rho e^{-(b-a)t}.$$

Incidentally, for this case the probability density of $u_r = s_r - t_r$ is

$$C'(u) = \begin{cases} ab(a+b)^{-1}e^{au} & (u < 0) \\ ab(a+b)^{-1}e^{-bu} & (u \geqq 0) \end{cases}.$$

Example b. Poisson Input, General Service. In this case (99) becomes

$$\frac{\beta(s)a}{a-s} - 1 = \frac{s - a + a\beta(s)}{a-s} = \frac{\psi_+(s)}{\psi_-(s)}.$$

With the help of Rouche's theorem (Titchmarsh,[15] p. 116) and the fact that $[1 - \beta(s)]/s$ is the Laplace transform of $1 - B(t)$, it may be shown that $s - a + a\beta(s)$ has no zero in $\text{Re}(s) > 0$ when $b > a$.

Then, with $D = a$,

$$\psi_+(s) = s - a + a\beta(s), \qquad \psi_-(s) = a - s,$$

$$K = (a^{-1} - b^{-1})\psi_-(0) = 1 - \rho = W(0), \tag{110}$$

$$w(s) = s\varphi_+(s) = \frac{sK}{\psi_+(s)} = \frac{s(1-\rho)}{[s - a + a\beta(s)]}.$$

The expression for $w(s)$ is sometimes called the Pollaczek-Khinchin formula. The coefficient of $-s$ in the power series expansion of $w(s)$ is equal to the average waiting time.

Example c. Recurrent Input, Exponential Service. Replacing $\beta(s)$ by $b/(b + s)$ in (99) gives

$$\frac{b\alpha(-s) - b - s}{b + s} = \left[\frac{b\alpha(-s) - b - s}{s(s - s_1)}\right]\left[\frac{s(s - s_1)}{b + s}\right] = \frac{\psi_+(s)}{\psi_-(s)}$$

where s_1 is the one and only zero of $s + b - b\alpha(-s)$ which lies in $\text{Re}(s) < 0$ when $b > a$. The existence of s_1 may be established with the help of Rouche's theorem.[15] If $s = 0$ is the only new zero of $s + b - b\alpha(-s)$ introduced when the region $\text{Re}(s) < 0$ is extended to $\text{Re}(s) < D$, one may take

$$\psi_+(s) = \frac{s(s - s_1)}{b + s}, \qquad \psi_-(s) = \frac{s(s - s_1)}{b\alpha(-s) - b - s}. \tag{111}$$

These functions satisfy the conditions stated just below (99). Then

$$K = \lim_{s\to 0} \frac{\psi_+(s)}{s} = -\frac{s_1}{b} = W(0)$$

$$\varphi_+(s) = \frac{K}{\psi_+(s)} = -\frac{s_1(b+s)}{bs(s-s_1)} = \frac{1}{s} - \frac{1+s_1b^{-1}}{s-s_1}$$

$$W(t) = 1 - (1+s_1b^{-1})e^{s_1t}.$$

Example d. Erlangian Input, Erlangian Service. In this case the input and service time probability densities are

$$A'(t) = \frac{al(alt)^{l-1}}{(l-1)!}e^{-alt}, \qquad B'(t) = \frac{bk(bkt)^{k-1}}{(k-1)!}e^{-bkt}$$

and have the Laplace transforms

$$\alpha(s) = \left(1+\frac{s}{al}\right)^{-l}, \qquad \beta(s) = \left(1+\frac{s}{bk}\right)^{-k}.$$

It is found that (99) becomes

$$\frac{1-f(s)}{f(s)} = \frac{\psi_+(s)}{\psi_-(s)}, \qquad f(s) = \left(1-\frac{s}{al}\right)^{l}\left(1+\frac{s}{bk}\right)^{k}.$$

For $a < b$ (the only case considered here), the polynomial $F(s) \equiv 1 - f(s)$ has a zero at the origin, zeros $s_1, \cdots, s_k$ in $\text{Re}(s) < 0$ and zeros $s_{k+1}, \cdots, s_{k+l-1}$ in $\text{Re}(s) > 0$. This may be shown with the help of Rouche's theorem. It turns out that $s_1, \cdots, s_k$ lie inside a circle of radius bk centered on $s = -bk$, and $s_{k+1}, \cdots, s_{k+l-1}$ lie inside a circle of radius al centered on $s = al$. Hence

$$\psi_+(s) = \frac{s(s-s_1)\cdots(s-s_k)}{(s+bk)^k},$$

$$\psi_-(s) = -\frac{(s-al)^l}{(s-s_{k+1})\cdots(s-s_{k+l-1})} \qquad (112)$$

$$W(0) = K = \frac{s_1\cdots s_k}{(-bk)^k} = \frac{(a^{-1}-b^{-1})(al)^l}{s_{k+1}\cdots s_{k+l-1}}.$$

For the case of regular arrivals and constant service time, inspection shows that $W(t) = 1$. It appears that in this case letting $k = l \to \infty$ should give

$$\psi_+(s) = s, \qquad \psi_-(s) = s[e^{sa^{-1}-sb^{-1}} - 1]^{-1}.$$

Example e. Arrival Rate Almost as Large as Service Rate. It often happens that when the arrival rate a is almost as large as the service rate b, most of the customers have to wait a long time for service. In such cases one may obtain an approximation for the waiting time distribution function $W(t)$.

Let a_1, a_2, b_1, b_2 be the first and second moments, and v_A, v_B be the variances (which are assumed to exist in the following discussion) associated with $A(t)$ and $B(t)$. Then $a_1 = 1/a$ is the average spacing between arrivals and $b_1 = 1/b$ is the average service time. From the integral (105) for $W(t)$ it is seen that the behavior of $W(t)$ for large values of t is determined by the right-most singularities of $K/\psi_+(s)$. In Example a (Poisson input, exponential service), these are poles at the zeros of $\psi_+(s)$ which occur at $s = 0$ and $s = s_1 = a - b$. Note that s_1 is negative and $s_1 \rightarrow 0$ as a tends to b. Furthermore, the value of $\psi_-(s)$ near the origin does not change markedly as $a \rightarrow b$; and the same is true for the remaining factor $(b + s)^{-1}$ appearing in $\psi_+(s)$.

Many other cases show the same type of behavior as $a \rightarrow b$. In general, the function (99)

$$\beta(s)\alpha(-s) - 1 = \frac{\psi_+(s)}{\psi_-(s)}$$

has a double zero at $s = 0$ when $a = b$. When a becomes slightly less than b, and a_1 slightly greater than b_1, one of these zeros remains at $s = 0$ and the other moves to $s \approx s_1$ where

$$s_1 = -\frac{2(a_1 - b_1)}{v_A + v_B} < 0. \tag{113}$$

This may be seen upon using

$$\begin{aligned} \beta(s) &= 1 - b_1 s + \frac{b_2 s^2}{2} + o(s^2) \\ \alpha(-s) &= 1 + a_1 s + \frac{a_2 s^2}{2} + o(s^2) \end{aligned} \tag{114}$$

and assuming that $(a_1 - b_1)^2$ is negligible in comparison with $v_A + v_B$. The approximation (113) for s_1 is given by Smith[14] who points out its importance in the present case.

An examination of the earlier examples leads us to take

$$\psi_+(s) \approx s(s - s_1)C \tag{115}$$

when s is near the origin. Here C is a constant equal to the value of the remaining portion of $\psi_+(s)$ at $s = 0$. Equations (103) and (105) then give

$$K = \lim_{s \to 0} \frac{\psi_+(s)}{s} = -s_1 C \tag{116}$$

$$\begin{aligned} \frac{K}{\psi_+(s)} &\approx \frac{-s_1}{s(s - s_1)} = \frac{1}{s} - \frac{1}{s - s_1} \\ W(t) &\approx 1 - e^{s_1 t}. \end{aligned} \tag{117}$$

Thus, when $a \rightarrow b$ and $v_A + v_B$ does not tend to zero, $W(t)$ is given by the approximation (117) where s_1 is given by (113). The average waiting time is $-1/s_1$. It should be noted that this approximation gives $W(0) \approx 0$ instead of the true (small) value $W(0) = K = -s_1 C$. Unfortunately, there appears to be no simple expression for C corresponding to (113) for s_1.

A.4 *Conclusion*

Finally, it will be mentioned that when a customer departs after being served, the chance p_n that n customers remain in the system is equal to the chance that an arriving customer will find n in the system. Furthermore, for $0 \leqq x < 1$

$$\sum_{n=0}^{\infty} x^n p_n = \frac{(1 - x)}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\alpha(-s)\beta(s)}{[1 - x\alpha(-s)]} \frac{K\, ds}{\psi_+(s)}$$

where $c > 0$ and is such that the singularities of $\beta(s)/\psi_+(s)$ and $\alpha(-s)/[1 - x\alpha(-s)]$ lie on opposite sides of the path of integration (this restricts $A(t)$ and $B(t)$ somewhat). The right-hand side may be replaced by one of several integrals which differ slightly from the one shown.

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