UNIVERSITY OF ILLINOIS

DIGITAL COMPUTER

LIBRARY ROUTINE V 5 - 184

TITLE	Spherical Bessel Functions
TYPE	Closed
NUMBER OF WORDS	59 (38 plus the 21 of Sine-Cosine Routine T - 5 which is attached.)
TEMPORARY STORAGE	0 - 6 inclusive and (k_e + 1) locations at S3
PRESET PARAMETER	S3 contains the address at which the table of functions j_0 (x), j_1(x), ... , j_{k_e} (x) is to be placed.
DURATION	(18 + 2.5 k_e) ms where k_e is the order of the highest Bessel Function to be found.
ACCURACY	Extremely variable and dictates that this program shall be used with care. See discussion under "Method and Error Discussion" below.

DESCRIPTION AND ENTRY

Enter with the scaled argument $2^{-n}x$ in the accumulator (A) and the program parameter:

p	50 (16k_e + n)F 50 pF	$1 \leq k_e \leq 15$
p + 1	26 qF	

where 2^{-n} is the scale factor on the argument, k_e is the maximum order of the Bessel Functions to be found, and q is the address of this program. On exit j_r(x) will be found in rS3 i.e. j_0(x) in S3, j_1(x) in 1S3, j_2 (x) in 2S3, ... , j_{k_e} (x) in k_eS3.

NOTE

This program contains Sine-Cosine Library Routine T 5 - 157 as an appended part. It is available for use in other connections, so it need not be placed a second time in the machine. Merely enter it at word 38L of this program.

METHOD AND ERROR DISCUSSION

The Spherical Bessel Functions $j_k(x)$ are defined in terms of the ordinary Bessel Functions

$j_k(x)$ by

$$j_k(x) = \sqrt{\frac{\pi}{2x}}\; J_{k+1/2}(x) \quad . \tag{1}$$

Explicitly:

$$j_0\ (x) = \frac{\sin x}{x} \quad , \tag{2}$$

$$j_1\ (x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \quad , \tag{3}$$

$$j_2\ (x) = \left(\frac{3}{x^3} - \frac{1}{x}\right) \sin x - \frac{3}{x^2} \cos x,$$

etc.

They satisfy the recurrence relation:

$$j_{k+1}(x) = \frac{(2k+1)}{x}\ j_k(x) - j_{k-1}(x). \tag{4}$$

Although the argument of $j_k(x)$ need not be less than 1 and hence in scale, $|j_k\ (x)| < 1$ and hence is in scale.

In this routine the sine-cousine routine is used to evaluate $j_0(x)$ and $j_1(x)$ using (2) and (3). The relation (4) is then used to construct the table of functions.

Although this method is simple from a programming point of view, since it obviates the tedious evaluation of series, it can be subject to large numerical error. This error in $j_k(x)$ may be as large as of the order of:

$$(2k - 1)!! \, x^{-k} \, 2^{-37} \qquad (5)$$

when $x < 2k + 1$, where

$(2k - 1)!! \equiv (2k + 1)\ (2k - 3) \ldots (5)\ (3)\ (1).$

When $x > 2k + 1$, the error is of the order of $k2^{-38}$. These estimates give the order of magnitude of the upper limits of the error. Table 1 gives the error as determined experimentally from consulting tables of these functions. This table could not be read to better than 1 part in 10^{11}.

For $x \geq 10$ and $j_k \leq 13$, no error could be detected using the number of significant figures available in existing tables.

All of these tests were made with an optimum choice of the scaling factor 2^{-n}, i.e. n as small as possible consistent with $2^{-n}x$ remaining in scale.

For a given argument, in the regions of k labeled "worthless" the answers lose significance to such an extent that the program starts to be subject to division hangup in the later loops in attempting to compute the higher j_k.

It should be noted that in the regions labeled "worthless" for a given argument, a good approximation to j_k is zero. This approximation will (in these regions) be in error by less than the entry just above the region. This is because (in order of magnitude):

$$j_k(.01) \leq 10^{-8} \quad \text{for } k \geq 3$$
$$j_k(.06) \leq 10^{-8} \quad \text{for } k \geq 4$$
$$j_k(0.1) \leq 10^{-7} \quad \text{for } k \geq 4$$
$$j_k(1.0) \leq 10^{-7} \quad \text{for } k \geq 8$$
$$j_k(1.5) \leq 10^{-7} \quad \text{for } k \geq 9$$
$$j_k(3.0) \leq 10^{-7} \quad \text{for } k \geq 12$$

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APPROVED BY JPNash

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DH : mge

TABLE 1

Error in j_k For Argument

Function	.01	.06	0.1	1.0	1.5	3.0	5.0	10.0
j_0	10^{-10}	10^{-11}	10^{-11}	$< 10^{-11}$	$< 10^{-11}$	$< 10^{-11}$	$< 10^{-11}$	$< 10^{-11}$
j_1	10^{-8}	5×10^{-10}	2×10^{-10}	$< 10^{-11}$	$< 10^{-11}$	$< 10^{-11}$	$< 10^{-11}$	$< 10^{-11}$
j_2	10^{-6}	2×10^{-8}	5×10^{-9}	$< 10^{-11}$	$< 10^{-11}$	$< 10^{-11}$	$< 10^{-11}$	$< 10^{-11}$
j_3	worthless	2×10^{-6}	3×10^{-7}	10^{-11}	10^{-11}	$< 10^{-11}$	$< 10^{-11}$	$< 10^{-11}$
j_4		worthless	worthless	7×10^{-11}	5×10^{-11}	$< 10^{-11}$	$< 10^{-11}$	$< 10^{-11}$
j_5				6×10^{-10}	4×10^{-10}	$< 10^{-11}$	$< 10^{-11}$	$< 10^{-11}$
j_6				10^{-8}	2×10^{-9}	10^{-11}	10^{-11}	$< 10^{-11}$
j_7				6×10^{-8}	3×10^{-9}	2×10^{-11}	2×10^{-11}	$< 10^{-10}$
j_8				worthless	8×10^{-7}	8×10^{-11}	7×10^{-11}	$< 10^{-10}$
j_9					worthless	4×10^{-10}	2×10^{-10}	$< 10^{-10}$
j_{10}						2×10^{-9}	8×10^{-10}	$< 10^{-9}$
j_{11}						2×10^{-8}	3×10^{-9}	$< 10^{-9}$
j_{12}						worthless	10^{-8}	$< 10^{-8}$
j_{13}							6×10^{-8}	$< 10^{-8}$

LOCATION	ORDER		NOTES
0	40 3F		$2^{-n} \rightarrow 3$
	K5 F		Plant Link
1	42 25L		
	10 24F		
2	42 10L		Plant k_e
	01 4F		n in R_1 (RHA)
3	42 6L		Plant n
	42 9L		
4	42 16L		
	42 19L		
5	42 28L		
	50 3F		$2^{-n}x$ in R_2
6	75 37L		$2^{-n} (\frac{x}{\pi})$ in R_1 R_2
	00 (n)F	By 3	$(x/_\pi)$ mod 2 in R_1
7	40 4F		$(\frac{x}{\pi})$ mod 2 $\rightarrow 4$
	50 7L		
8	26 38L		2^{-1} sin x in R_1
	00 1F		sin x in R_1
9	40 5F		sin x $\rightarrow 5$
	10 (n)F	By 3	2^{-n} sin x in R_1 R_2
10	66 3F		j_0 in R_2
	S5 (k_e)F	By 2	j_0 in R_1 (End const.)
11	50 3F		x^{-n} in R_2
	40 S3		$j_0 \rightarrow S3$

LOCATION	ORDER		NOTES PAGE 2
12	7J 3F		x^{-2n} in R_1
	40 6F		x^{-2n} in 6
13	LJ 4F		$\frac{x}{\pi} + 1/2$ in R_1
	50 13L		
14	26 38L		2^{-1} cos x in R_1
	00 1F		Cos x in R_1
15	40 4F		Cos x $\rightarrow$ 4
	4S 31L		Clear k counter
16	L5 5F		sin x in R_1
	10 (n) F	By 4	2^{-n} sin x
17	40 F		(2^{-n} Sin x) (msp) $\rightarrow$ 0
	S5 F		(2^{-n} Sin x) (ksp) $\rightarrow R_1$
18	50 3F		
	70 4F		-2^{-n}x cos x in R_1 R_2
19	L4 F		2^{-n} (sin x - x cos x) in R_1 R_2
	10 (n)F	By 4	2^{-n} (sin x - x cos x) in R_1 R_2
20	66 6F		j_1 in R_2
	S5 F		j_1 in R_1
21	40 1S3		$j_1 \rightarrow$ 1S3
	L5 11L		address S3, 1S3, and 2S3
22	42 32L	From 35	} advance
	F4 36L		+ 1 } addresses
23	42 29L		}
	F4 36L		+ 1 }

LOCATION	ORDER		NOTES
24	42 34L		
	F5 31L		$k\ 2^{-39}$ in R_1
25	L0 10L		$(k - k_e)\ 2^{-39}$ in R_1
	32 ( )F	By 1	⊕ Done - out by link
26	L4 10L		⊖ Not done
	40 31L		$k \times 2^{-39}$ in R_1
27	50 36L		Clear R_2
	00 1F		$2k\ 2^{-39}$
28	F4 36L		$(2k + 1)\ 2^{-39}$
	10 (n)F	By 5	$(2k + 1)\ 2^{-n}$ in R_2
29	10 4F		$(2k + 1)\ 2^{-n-4}$ in R_2
	75 (1S3)F	By 23	$(2k + 1)\ j_1\ (2^{-n-4})$ in $R_1\ R_2$
30	00 3F		$(2k + 1)\ j_1\ 2^{-n-1}$ in $R_1\ R_2$
	36 31L		Waste
31	66 3F		$(2k + 1)\ \frac{j_1}{2x}$ in R_2
	S5 ( )F		and in R_1 (Counter)
32	40 F		$(2k + 1)\ \frac{j_1}{2x}$ in 0
	L1 (S3)F	By 22	$- j_0$
33	10 1F		$- j_0\ 2^{-1}$
	L4 F		$j^2/2$
34	00 1F		j_2 in R_1
	40 (2S3)F	By 24	$j_2 \rightarrow$ 2S 3
35	F5 32L		Step addresses by 1
	26 22L		and repeat
36	00 F		$\doteq$ 0
	00 F		

LOCATION	ORDER	NOTES	PAGE 4
37	00 F	$\doteq \frac{1}{\pi}$	
	00 318 309 886 184J		
38	50 39L	Interlude to input Routine T5	
	26 999F		
39	00 F		
	00 38L		
40	00 F		
	26 38L		
41	26 1N		
42	Routine T5 (Sine Routine) ends this tape. It will be loaded at 38L.		

V5-184

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