UNIVERSITY OF ILLINOIS

DIGITAL COMPUTER

LIBRARY ROUTINE M 13 - 179

TITLE — Complete Linear Matrix Equation Solver and General Matrix Inversion

TYPE — Complete program

DURATION — n = order of matrix A, m = number of columns of matrix B.

a) reading in of matrix A and B: read-in speed (250 characters per second)

b) computing time: $n^3/500$ (m = 1)

: $-11 + 4n - .3n^2 + .01n^3$ (n = m)

c) punching of matrix X: punch speed (60 characters per second)

ACCURACY — A function of the order and conditioning of the matrices.

DESCRIPTION — This routine solves the linear matrix equation AX = B, for matrices A and B satisfying the conditions:

a) A is non-singular and of size (nxn)

b) B is of size (nxm)

where the magnitude of n and m is governed by the inequality:

$$nm + n/2(n + 1) + (n + m) \leq 842$$

The solution X, of size (nxm), is punched out by successive columns. In the special case when B = I (the identity matrix), we obtain the inverse since the solution is $X = A^{-1}$. Here n = m and the limits on the size of A are $n \leq 22$.

DATA TAPE PREPARATION — A distinction is to be made between the cases when B = I (i.e. when we wish to invert A) and $B \neq I$ (i.e. when we are in essence solving m sets of n simultaneous equations in n unknowns).

1) B = I Punch the matrix A, row by row, adhering to the following:

a) Each row is ended by J.

b) Follow the final J with a sexadecimal character 1, 2, ..., K, S indicating the number of digits desired in the punching of X.

2) $B \neq I$: Punch the augmented matrix [A,B] row by row, such that:

   a) Each of the respective rows of A and B are ended by N.

   b) Follow the final N of the last row of B by a sexadecimal character 1, 2, ..., K, S indicating the number of digits desired in the punching of the output X.

3) In either case:

   a) All rows must be so scaled that no element is $\geq 1/2$.

   b) Each element must be preceded by a plus or minus sign and may contain up to 12 figures. (Indicate a plus sign for zero).

<u>Example</u>: A typical input tape for $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -4 & 1 \\ -2 & 5 & 6 \end{bmatrix}$

and $B = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ -2 & 0 \end{bmatrix}$

would appear as follows:

```
+1   +2   +    N     +1   +2  N
+3   -4   +1   N     +    -1  N
-02 +05 +06    N     -02  +   N K
```

and will print out to 10 places rounded off. If B were I we would punch instead:

```
+01 +02 +    J
+03 -04 +01  J
-02 +05 +06  J K
```

Note: Concern must be given to the scaling possibilities offered by each case. In the former case, each row can be considered independently of the others, remembering, however, that corresponding rows of A and B must be treated as a unit. In the latter case, scaling one row forces one to scale each row.

### METHOD OF USE

1) Read in the routine until it stops.
2) Place the data tape into the reader and start by raising the black switch.
3) The computation is then carried out and subsequently X will be punched out. The machine will then stop on a transfer order (24 52) and is then ready to accept another equation for solution upon raising the black switch. No concern need be given to successive values of matrix sizes.

### DATA OUTPUT

Each column of X will be punched out in succession, followed by a scaling factor and the character N. This scaling factor may be different for each column. It indicates the position of the decimal point as lying after the column in which the factor 1 appears. A typical output for a 3 x 3 matrix may look like the following:

```
+1000
-0500
+2500
+1000N
-2000
+0030
-2500
+0100N
+0300
+0100
+0050
+1000N
```

representing

$$\begin{bmatrix} 1 & -20 & .3 \\ -.5 & .3 & .1 \\ 2.5 & -25 & .05 \end{bmatrix}$$

Successive calculations are separated by three carriage returns and line feeds.

Note: In case we are inverting A(i.e., B = I), then the output matrix must be multiplied by $10^{n+1}$ if the original matrix A was scaled by $10^n$.

### MATHEMATICAL METHOD

1) The matrix A is upper triangularized by a series of elementary row operations, $\prod E_i$, during which time the same operations are performed upon the augmented matrix B, obtaining thereby $(\prod E_i)\, B = B'$.
2) Successive columns of B' are then considered and by the back substitution method we solve m sets of n simultaneous equations in n unknowns. The m

column vectors comprising this solution form the matrix X.

3) For the case B = I, it follows that $X = A^{-1}$.

4) In the event a zero appears on the diagonal of the upper triangular matrix, the machine detects the singularity of A and prints out F, indicating that there is no solution. It is then ready to proceed to the next problem.

5) Appearance of a zero on the diagonal of the upper triangularized matrix is a necessary condition for singularity of A. This is not true, however, for the machine representation, due to accumulating round off errors. In the process of upper triangularizing A, automatic scaling is performed so that the absolute value of the largest element in each row of the augmented matrix lies in the interval (1/2, 1/4]. This destroys the possibility of comparing the relative size of the diagonal elements upon the completion of the calculation, in order to obtain an indication of the rank of the matrix. It is, therefore, impossible to guarantee detection of a nearly singular matrix. However, it should be noted that the size of the scaling factors appearing for each column of X will indicate the conditioning of A. In the event that scaling gets beyond the capacity of the registers of Illiac the calculation is stopped and F is printed out. The machine is then ready for another calculation.

6) In addition to the precautions of this routine, it is advisable, in questionable instances, to form the product of A and the X, computed by this routine, and note the approximation to the matrix B. Another alternative would be to employ Routine M 12 by which one can find the determinant of A and can inspect the elements on the diagonal of the upper

triangularized A and obtain the rank.

NOTES

1) The triangularization process and back substitution is a modification of Routine L 1.

2) Since $(A^T)^{-1} = (A^{-1})^T$ one can either punch rows of $A^{-1}$ or columns depending on whether the input matrix is $A^T$ or A respectively.

3) For the case B = I, the identity matrix is automatically augmented within the routine.

4) A sample inversion of a matrix of order 22 required approximately 35 seconds computing time. Errors were less than $10^{-11}$.

5) A sample solution of a system of 38 equations in 38 unknowns with random integer coefficients required approximately 100 seconds computing time. Errors were less than $10^{-10}$.

DATE May 4, 1955
CODED BY M. L. Frank
APPROVED BY J. P. Nash

| LOCATION | ORDER | | NOTES PAGE 1 M 13 |
|---|---|---|---|
| | Routine X 1 | | Decimal Order Input |
| | 00 8K | | |
| | Routine N 2 | | Input a Sequence of Decimal Fractions |
| | 00 34K | | |
| | Routine P 2 | | Print (A) with or without Sign to n Places |
| | 00 52K | | |
| 0 | 41 7F | | Clear row counter |
| | 41 6F | | Clear counter |
| 1 | 50 183F | | |
| | 50 1L | | Read in row r of A |
| 2 | 26 8F | | |
| | 40 F | | Store 0 or 1 |
| 3 | L5 16F | | 183 + n = y (left address) |
| | 46 17L | | |
| 4 | 46 83L | | |
| | 46 91L | | |
| 5 | 10 20F | | |
| | 42 82L | | Set address of y |
| 6 | 42 99L | | |
| | 42 10L | | |
| 7 | 42 126L | | |
| | 42 129L | | |
| 8 | L0 46L | | y - 183 |
| | 42 4F | | Store m = n |
| 9 | 42 3F | | Store n |
| | L3 F | | Test for inversion or solving AX = B |
| 10 | 30 17L | | |
| | 41 (y)F | 6' 11' | |
| 11 | F5 10L | | |
| | 42 10L | | Clear y to (y + n - 1) |
| 12 | L0 3F | | |
| | L0 126L | | |
| 13 | 32 10L | | |
| | L5 126L | | |
| 14 | L4 7F | | y + r |
| | 42 15L | | Augment unit matrix |
| 15 | L5 125L | | 1/10 |
| | 40 (y+r)F | 14' | |

| LOCATION | ORDER | | NOTES |
|---|---|---|---|
| 16 | 22 20L | | |
| | 00 F | | Waste |
| 17 | 50 (y)F | 3' | Read in row r of B if B ≠ I |
| | 50 17L | | |
| 18 | 26 8F | | |
| | L5 16F | | y + m = t |
| 19 | 10 20F | | |
| | L0 126L | | t - y = m |
| 20 | 42 4F | | Store m |
| | L5 48L | | |
| 21 | 42 26L | | |
| | L5 4F | | 183 + n + m = t |
| 22 | L4 126L | | |
| | 42 48L | | |
| 23 | 42 27L | | |
| | 42 128L | | |
| 24 | 00 20F | | |
| | 46 127L | | |
| 25 | 22 25L | | Waste |
| | L3 7F | | Cause ith row to interchange with virtual ith row |
| 26 | 32 27L | | |
| | 41 ( )F | 21 | |
| 27 | 26 28L | | |
| | 41 (t)F | 23 | |
| 28 | L5 48L | | Set addresses |
| | 42 38L | | |
| 29 | 42 41L | | |
| | 42 44L | | |
| 30 | 42 54L | | |
| | L5 46L | | |
| 31 | L4 6F | | 183 + i = x |
| | 42 47L | | |
| 32 | 42 51L | | |
| | 42 68L | | |
| 33 | 42 70L | | |
| | 42 42L | | |

| LOCATION | ORDER | | NOTES |
|---|---|---|---|
| 34 | 00 20F | | |
| | 46 41L | | |
| 35 | 46 39L | | |
| | 46 45L | | |
| 36 | 46 51L | | |
| | 46 53L | | |
| 37 | L5 123L | | |
| | 46 48L | | Prepare for interchange of rows |
| 38 | 46 49L | | |
| | L3 (t)F | 28' | Test size of leading elements |
| 39 | L6 (x)F | 35 | |
| | 32 42L | | |
| 40 | 47 48L | | No row interchange |
| | 50 7F | | Approximate zero in Q |
| 41 | L5 (x)F | 34' | |
| | 66 (t)F | 29 | |
| 42 | 26 46L | | |
| | 50 (x)F | 33' | |
| 43 | S3 F | | |
| | 32 45L | | |
| 44 | 50 124L | | $1 - 2^{-39}$ |
| | 75 (t)F | 29' | |
| 45 | 66 (x)F | 35' | |
| | 47 49L | | Row interchange |
| 46 | 41 5F | | Clear counter |
| | S1 183F | | Address a constant |
| 47 | 40 2F | | |
| | L5 (x)F | 31' 58' | |
| 48 | 40 ( )F | 37' 40 | |
| | L5 (t)F | 22' 56 | |
| 49 | 40 ( )F | 38 45' | |
| | 50 2F | | |
| 50 | 7J 1F | | |
| | L4 F | | |
| 51 | 40 (x)F | 36 57' | |
| | L3 (x)F | 32 57' | |

| LOCATION | ORDER | | NOTES |
|---|---|---|---|
| 52 | L6 5F | | $\lvert N(5)\rvert - \lvert N(x)\rvert$ Element of largest absolute value in row i; Linearly combine successive rows so as to get zeros below the diagonal of A |
| | 36 54L | | |
| 53 | L7 (x)F | 36' 58 | |
| | 40 5F | | |
| 54 | L5 1F | | |
| | 40 (t)F | 30 55' | |
| 55 | F5 54L | | |
| | 42 54L | | |
| 56 | 42 48L | | |
| | L5 51L | | |
| 57 | L4 123L | | |
| | 40 51L | | |
| 58 | 46 53L | | |
| | 42 47L | | |
| 59 | L0 127L | | |
| | 32 47L | | |
| 60 | L5 74L | | Reset order |
| | 40 69L | | |
| 61 | L3 5F | | If zero do not rescale |
| | 36 75L | | |
| 62 | LL 5F | | If absolute value of any one element of row is $\geq 1/2$ prepare order to scale down |
| | 32 65L | | |
| 63 | L5 70L | | |
| | 46 69L | | |
| 64 | 26 68L | | |
| | F5 69L | | |
| 65 | 42 69L | | If absolute value of largest element $< 1/2$ prepare to scale up rows so that this element is in the interval (1/2, 1/4] |
| | F5 69L | | |
| 66 | 42 69L | | |
| | L5 5F | | |
| 67 | 00 1F | | |
| | 40 5F | | |
| 68 | 50 7F | | Put ~ zero in Q |
| | L5 (x)F | 72 32' | |
| 69 | 10 (1)F | 63' 60' | |
| | 00 (1)F | 66 65 60' | |

| LOCATION | ORDER | | NOTES |
|---|---|---|---|
| 70 | 50 2F | | Waste, address a constant — Rescale row i |
| | 40 (x)F | 71' 33 | |
| 71 | F5 70L | | |
| | 42 70L | | |
| 72 | 42 68L | | |
| | L0 128L | | Test |
| 73 | 36 68L | | |
| | 26 75L | | |
| 74 | 10 1F | | Constant |
| | 00 1F | | |
| 75 | F5 6F | | Determine if row i must have further eliminations (done i times) |
| | 40 6F | | |
| 76 | L5 7F | | |
| | L0 6F | | |
| 77 | 36 28L | | |
| | F5 7F | | Count number of rows |
| 78 | 40 7F | | |
| | L0 3F | | |
| 79 | 36 80L | | |
| | 22 L | | Repeat for next row |
| 80 | 41 5F | | Clear counter |
| | 81 4F | | Read in character to determine number of digits to punch |
| 81 | 00 20F | | |
| | 46 115L | | |
| 82 | L5 125L | | Set scaling factor |
| | 40 (y)F | 5' | |
| 83 | L3 (y)F | 4 | Terminate calculation if scaling $< 2^{-39}$ |
| | 32 121L | | |
| 84 | 41 6F | | Clear counters |
| | 41 7F | | |
| 85 | L5 46L | | Set addresses |
| | 42 114L | | |
| 86 | L5 48L | | |
| | 42 96L | | |
| 87 | L0 4F | | |
| | L4 5F | | |
| 88 | 42 91L | | |
| | L5 96L | | |

| LOCATION | ORDER | | NOTES |
|---|---|---|---|
| 89 | L0 4F | | |
| | 42 96L | | |
| 90 | L5 91L | | |
| | 46 96L | | |
| 91 | 50 (y)F | 4' | Calculate $\sum_{j=i+1}^{n} a_{ij}x_j$ |
| | 71 ( )F | 88 94 | |
| 92 | 40 F | | |
| | L5 91L | | |
| 93 | F0 7F | | |
| | L0 4F | | |
| 94 | 42 91L | | |
| | 22 100L | | |
| 95 | 22 95L | | Waste |
| | S5 F | | |
| 96 | 50 (y)F | 90' 101' | |
| | 74 ( )F | 86' 89' 101' | |
| 97 | L4 F | | |
| | 40 F | | |
| 98 | LL F | | $\geq 1/2$? |
| | 32 100L | | |
| 99 | 50 125L | | Rescale and start again |
| | 7J (y)F | 6 | |
| 100 | 22 82L | | |
| | L5 96L | | Set addresses |
| 101 | L0 123L | | |
| | 40 96L | | |
| 102 | 42 106L | | |
| | 42 109L | | |
| 103 | 46 111L | | |
| | F5 6F | | Count (n - i + 1) times for row i |
| 104 | 40 6F | | |
| | L5 7F | | |
| 105 | L0 6F | | |
| | 32 95L | | |
| 106 | 41 6F | | Reset |
| | L3 $(a_{ii})$F | 102 | |

| LOCATION | ORDER | | NOTES |
|---|---|---|---|
| 107 | 32 121L | | End if zero on diagonal |
| | L6 F | | Test if division is proper |
| 108 | 36 99L | | |
| | 26 109L | | Waste |
| 109 | L5 F | | |
| | 66 $(a_{ii})$F | 102' | |
| 110 | 22 110L | | Waste |
| | S1 F | | |
| 111 | 40 (y-i)F | 103 | |
| | F5 7F | | Count n rows |
| 112 | 40 7F | | |
| | L0 3F | | |
| 113 | 36 114L | | |
| | 22 88L | | Repeat |
| 114 | 92 131F | | Line feed (Punch out column j of X) |
| | L5 (x)F | 85' 117 | |
| 115 | 50 ( )F | 81' | To enter P 2 |
| | 50 115L | | |
| 116 | 26 34F | | |
| | F5 114L | | |
| 117 | 42 114L | | |
| | F0 129L | | |
| 118 | 36 114L | | |
| | 92 770F | | Punch N |
| 119 | F5 5F | | Count m columns |
| | 40 5F | | |
| 120 | L0 4F | | |
| | 36 122L | | |
| 121 | 26 82L | | Repeat |
| | 92 898F | | Punch F |
| 122 | 92 139F | | Three line feeds between problems |
| | 24 L | | Ready for next calculation |
| 123 | 00 1F | | |
| | 00 1F | | |
| 124 | 7L 4095F | | $1 - 2^{-39}$ |
| | LL 4095F | | |

| LOCATION | ORDER | | NOTES |
|---|---|---|---|
| 125 | 00 F | | 1/10 |
| | 00 1000 0000 0000 J | | |
| 126 | S6 17L | | |
| | 41 (y)F | 7 | |
| 127 | NO (t)F | 24' | |
| | L3 F | | End constants |
| 128 | J0 2F | | |
| | 40 (t)F | 23' | |
| 129 | 12 131F | | |
| | L5 (y)F | 7' | |
| | 24 52N | | |