# A SYMMETRIC CHAIN DECOMPOSITION OF L(4,n)

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Abstract.

$L(m,n)$ is the set of integer m-tuples $(a_1,\ldots,a_m)$ with $0 \le a_1 \le \ldots \le a_m \le n$, ordered by $a \le b$ when $a_i \le b_i$ for all $i$. R. Stanley conjectured that $L(m,n)$ is a symmetric chain order for all $(m,n)$. We verify this by construction for $m = 4$.

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$L(m,n)$ is defined as the lattice formed by order ideals in the direct product of two chains with m and n elements, respectively. Equivalently, it is the collection of integer sequences $a = (a_1,\ldots,a_m)$ satisfying $0 \le a_1 \le \ldots \le a_m \le n$, withordering $\underline{a} \le \underline{b}$ when $a_i \le b_i$ for all i. The correspondence is simple. If the chain elements are $x_1 < \ldots < x_m$ and $y_1 < \ldots < y_n$, then the number of elements paired with $x_i$ in the ideal corresponding to a is $n-a_i$. In other words, the antichain generating the ideal is $\{(x_1,y_{n-a_1}),\ldots,(x_m,y_{n-a_m})\}$.

Clearly, the rank of element a is $\sum a_i$, the rank of the entire lattice is mn, and the cardinality of the lattice is $\binom{m+n}{m}$. For any element a, we define its conjugate $a^* = (n-a_m,\ldots,n-a_1)$. Note that $a^{**} = a$. The ranks of an element and its conjugate sum to mn, so the sizes of the ranks are symmetric about the middle. Using complex algebraic methods, R. Stanley [3] proved the sizes of the ranks are also unimodal. These are necessary conditions for a stronger property he conjectured also holds. He conjectured that $L(m,n)$ is a symmetric chain order. A symmetric chain order is one whose elements can be partitioned into chains which are saturated (skip no ranks) and symmetric about the middle rank. The conjecture is clear when $m = 1$ or $m = 2$. Lindström [2] provided an inductive construction to verify it for $m = 3$. Here we give a construction somewhat different from his which verifies the conjecture when $m = 4$.

Let $S(m,n)$, the "shell" of $L(m,n)$, be those elements which begin with 0 or end with n. When these are removed from $L(m,n)$ the remainder is isomorphic to $L(m,n-2)$. The conjecture holds trivially when $n = 1$, and $L(m;0)$ can be defined as having a single element.

So, providing a symmetric chain decomposition of S(m,n) proves the conjecture by induction. We use this approach here for L(4,n) . Unfortunately, when m is odd and n is even the rank sizes in S(m,n) are not unimodal. So, for that case Lindström was forced to strip off two shells for his induction. For m = 4 this difficulty does not arise. It is possible that Lindström's construction generalizes for odd m and this does so for even m . When m and n both exceed 2 , L(m,n) is not an LYM-order, so Griggs' sufficient conditions for a symmetric chain order [1] cannot be applied.

<u>Theorem.</u> L(4,n) is a symmetric chain order.

It suffices to give a symmetric chain decomposition of S(4,n) . The chains will be of two types, $C_{ij}$ and $D_{ij}$ for suitable values of i and j. The chains are clearly saturated, so two steps will complete the proof.

(1) No element appears in more than one chain.

(2) The number of elements in the construction is the size of S(m,n) .

Each chain is composed of six segments, with the top element of one segment and the bottom element of the next identical. Throughout a given segment only one position in the integer sequence changes. Table 1 explicitly defines the chains and gives the ranks where the changes between segments occur.

Segments must have length at least 0 . That is, top and bottom elements may be identical, but the top element must not have rank below the bottom element. Examining the lengths of segments and ensuring that

| rank | $C_{ij}$ | segment | $D_{ij}$ | rank |
|---|---|---|---|---|
| 4n-6i-2j | n-3i-j , n-2i-j , n-i , n) | | (n-3i-j-2 , n-2i-j-1 , n-i , n) | 4n-6i-2j-3 |
| | ⋮ | 6 | ⋮ | |
| 4n-6i-3j | n-3i-j , n-2i-j , n-i-j , n) | | (j+1 , n-2i-j-1 , n-i , n) | 3n-3i |
| | ⋮ | 5 | ⋮ | |
| 3n-3i-2j | 0 , n-2i-j , n-i-j , n) | | (j+1 , i+j+1 , n-i , n) | 2n+2j+2 |
| | ⋮ | 4 | ⋮ | |
| 3n-3i-3j | (0 , n-2i-j , n-i-j , n-j) | | (j+1 , i+j+1 , 2i+j+1 , n) | n+3i+3j+3 |
| | ⋮ | 3 | ⋮ | |
| 2n-2j | (0 , i , n-i-j , n-j) | | 0 , i+j+1 , 2i+j+1 , n) | n+3i+2j+2 |
| | ⋮ | 2 | ⋮ | |
| n+3i | (0 , i , 2i+j , n-j) | | (0 , i+j+1 , 2i+j+1 , 3i+j+1) | 6i+3j+3 |
| | ⋮ | 1 | ⋮ | |
| 6i+2j | (0 , i , 2i+j , 3i+j) | | (0 , i+1 , 2i+j+1 , 3i+j+1 | 6i+2j+3 |

Table 1

we have legal elements at the bottom of $C_{ij}$ and the top of $D_{ij}$ yields necessary conditions on i and j . We claim the desired decomposition is obtained by taking all chains for which these necessary conditions are satisfied.

$$S(4,n) = \{C_{ij}: 3i+2j \le n, i \ge 0, j \ge 0\} \cup \{D_{ij}: 3i+2j \le n-3, i \ge 0, j \ge 0\} .$$

Figure 1 gives $S(4,7)$ explicitly as an example.

| $C_{00}$ | $C_{01}$ | $C_{02}$ | $C_{03}$ | $D_{00}$ | $D_{01}$ | $D_{02}$ | $C_{10}$ | $C_{11}$ | $C_{12}$ | $D_{10}$ | $C_{20}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 7777 | 6677 | 5577 | 4477 | 5677 | 4577 | 3477 | 4567 | 3467 | 2367 | 2467 | 1357 |
| 6777 | 6667 | 5567 | 4467 | 4677 | 3577 | 3377 | 3567 | 3457 | 2357 | 1467 | 0357 |
| 5777 | 5667 | 5557 | 4457 | 3677 | 2577 | 3367 | 2567 | 2457 | 2347 | 1367 | 0257 |
| 4777 | 4667 | 4557 | 4447 | 2677 | 2477 | 3357 | 1567 | 1457 | 1347 | 1267 | 0247 |
| 3777 | 3667 | 3557 | 3447 | 1677 | 2377 | 3347 | 0567 | 0457 | 0347 | 1257 | 0246 |
| 2777 | 2667 | 2557 | 2447 | 1577 | 2277 | 3337 | 0467 | 0456 | 0346 | 1247 | |
| 1777 | 1667 | 1557 | 1447 | 1477 | 2267 | 2337 | 0367 | 0356 | 0345 | 1237 | |
| 0777 | 0667 | 0557 | 0447 | 1377 | 2257 | 1337 | 0267 | 0256 | 0245 | 0237 | |
| 0677 | 0666 | 0556 | 0446 | 1277 | 2247 | 0337 | 0167 | 0156 | 0145 | 0236 | |
| 0577 | 0566 | 0555 | 0445 | 1177 | 2237 | 0336 | 0157 | 0146 | | 0235 | |
| 0477 | 0466 | 0455 | 0444 | 1167 | 2227 | 0335 | 0147 | 0136 | | 0234 | |
| 0377 | 0366 | 0355 | 0344 | 1157 | 1227 | 0334 | 0137 | 0135 | | | |
| 0277 | 0266 | 0255 | 0244 | 1147 | 0227 | 0333 | 0127 | 0134 | | | |
| 0177 | 0166 | 0155 | 0144 | J-J-37 | 0276 | 0233 | 0126 | | | | |
| 0077 | 0066 | 0055 | 0044 | 1127 | 0225 | 0133 | 0125 | | | | |
| 0067 | 0056 | 0045 | 0034 | 1117 | 0224 | | 0124 | | | | |
| 0057 | 0046 | 0035 | 0033 | 0117 | 0223 | | 0123 | | | | |
| 0047 | 0036 | 0025 | | 0116 | 0222 | | | | | | |
| 0037 | 0026 | 0024 | | 0115 | 0122 | | | | | | |
| 0027 | 0016 | 0023 | | 0114 | | | | | | | |
| 0017 | 0015 | 0022 | | 0113 | | | | | | | |
| 0007 | 0014 | | | 0112 | | | | | | | |
| 0006 | 0013 | | | 0111 | | | | | | | |
| 0005 | 0012 | | | | | | | | | | |
| 0004 | 0011 | | | | | | | | | | |
| 0003 | | | | | | | | | | | |
| 0002 | | | | | | | | | | | |
| 0001 | | | | | | | | | | | |
| 0000 | | | | | | | | | | | |

Figure 1. $S(4,7)$

Outline of Proof. To show the elements are all distinct, we express the D-chains in terms of the C-chains and then restrict our attention to the C -chains. Let $C_{ij}^{r}$ be the element of $C_{ij}$ of rank $r$, similarly for $D_{ij}^{r}$. We claim that chain $D_{i,j-1}$ is the conjugate of chain $C_{i,j}$ when the top and bottoms elements of the latter are removed. That is, $(D_{i,j-1}^{r})^{*} = C_{i,j}^{4n-r}$. It suffices to perform the conjugation on the transition elements between segments of $D_{i,j-1}$. They become the transition elements of $C_{i,j}$. Note the top and bottom elements of $C_{i,j}$ are unaffected and are conjugates of each other. Whenever $D_{i,j-1}$ exists, $C_{ij}$ exists. The affected $C_{ij}$ are those where $j>0$ and $3i+2j < n$.

Distinctness now reduces to showing:

(1a) The elements of $\cup\{C_{ij}\}$ are all distinct.

(1b) The chains $C_{i0}$ and $C_{i,(n-3i)/2}$ are self-conjugate.

(1c) There are no conjugate pairs among the elements of $\cup\{C_{ij}\}$, where $0 < j < (n-3i)/2$, other than the tops and bottoms of chains.

(1b) is seen immediately by conjugating the transition elements in those chains. The other two statements require eliminating a large number of easy cases.

To show we have the correct number of elements, we proceed by induction. Simple counting verifies it for small $n$. In general, the size of $S(m,n)$ is $|L(m,n)| - |L(m,n-2)|$. So,

$$|S(4,n)| = \binom{n+4}{4} - \binom{n+2}{4} = \frac{(n+1)(n+2)(2n+3)}{6}$$

This is the sum of a familiar sequence. Indeed,

$$|S(4,n)| - |S(4,n-1)| = (n+1)^2 .$$

Now we examine the changes in the construction between n-l and n , For all values of i and j such that $C_{ij}$ or $D_{ij}$ exists in the construction for n-l , a similarly indexed chain exists in the construction for n . Subtracting ranks, the number of elements in $C_{ij}$ is $4(n-3i-j)+1$ , and the number in $D_{ij}$ is $4(n-3i-j)-5$ . Each of these chains has 4 more elements than the similarly indexed chain in $S(4,n-1)$ , if that chain exists. We will see there is a $C_{ij}$ for every element of the middle rank which begins with 0 and a $D_{ij}$ for every such element whose first position is not zero.

The chains which arise newly when n is reached are those $C_{ij}$ for which $3i+2j = n$ and those $D_{ij}$ for which $3i+2j = n-3$ . For each value of i from 0 up to $\lfloor n/3 \rfloor$ or $\lfloor n/3 \rfloor -1$ , depending on parities, there will be one new $C_{ij}$ or $D_{ij}$ but not both.

Verifying that the construction picks up the proper number of elements reduces to:

(2a) Computing (and multiplying by 4) the number of chains in the construction for $S(4,n-1)$ -- that is, the sum of the number of solutions to $3i+2j < n-1$ and $3i+2j \le n-4$ .

(2b) Computing the total number of elements in new chains.

(2c) Verifying the sum of new elements in (2a) and (2b) is $(n+1)^2$ .

(2b) breaks into cases depending on the parity of n , and (2a) does the same with the parity of $\lfloor n/3 \rfloor$ , so (2c) requires six cases, depending on the congruence class of n modulo 6 .

<u>Details of Step 1</u>. If (la) does not hold, suppose $a = C^r_{ij} = C^r_{k\ell}$ . We have a number of cases to consider, depending on which segment contains a

in each of the two chains. Let ${}^{p}C_{ij}$ denote segment $p$ in $C_{ij}$. Equating the descriptions of the segments in Table 1 give us a number of linear relationships between $i$, $j$, $k$, and $\ell$. If $\underline{a}$ comes from ${}^{p}C_{ij}$ and ${}^{p}C_{k\ell}$, equating the positions which do not change in that segment implies $i = k$ and $j = \ell$ in all six cases, by straightforward subtraction of equalities.

By symmetry we may assume $\underline{a}$ occurs in a lower numbered segment in $C_{ij}$ than in $C_{k\ell}$. We allow the transition elements between segments to belong to either segment. So, if $\underline{a}$ is in ${}^{p}C_{ij}$ and ${}^{q}C_{k\ell}$, we may assume $\underline{a}$ is not the top element of ${}^{p}C_{ij}$ nor the bottom element of ${}^{q}C_{k\ell}$, else we have a case with smaller $q-p$. In particular, the rank of the top element in ${}^{p}C_{ij}$ must be strictly greater than the rank of the bottom element in ${}^{q}C_{k\ell}$.

Suppose $q = p+1$. This comparison of ranks yields a strict inequality when a particular linear function is applied to $(i,j)$ and to $(k,\ell)$. Whenever $q = p+1$ two positions in the elements remain constant from the bottom of segment $p$ to the top of segment $q$. This expresses two positions of $\underline{a}$ as identical linear functions of $(i,j)$ and $(k,\ell)$. In all five cases, we readily get the same linear function we obtained by considering ranks, but with equality this time.

If the first position of $\underline{a}$ is nonzero, $\underline{a}$ can occur only in segments 5 or 6. If it is zero, $\underline{a}$ occurs in segment 4 or below. This eliminates all but three of the cases which might have $C^{r}_{ij} = C^{r}_{k\ell}$ with $(i,j) \neq (k,\ell)$. The remainder we handle individually.

If $\underline{a}$ is in ${}^{2}C_{ij}$ and ${}^{4}C_{k\ell}$, positions 2 and 3 require $i = n-2k-\ell$ and $n-i-j > n-k-\ell$. Adding these gives $n-j > 2n-3k-2\ell \geq n$.

Next suppose $\underline{a}$ is in ${}^1C_{ij}$ and ${}^3C_{k\ell}$ . Equality of the last three positions requires $k < i$ , $n-k-\ell = 2i+j$ , and $n-\ell \geq 3i+j$ . Substituting for k and $n-\ell$ in the equation gives $2i+j < 2i+j$ . Finally, suppose $\underline{a}$ is in ${}^1C_{ij}$ and ${}^4C_{k\ell}$ . Comparing the top of ${}^1C_{ij}$ with the bottom of ${}^4C_{k\ell}$ yields $n+3i > 3n-3k-3\ell \geq n+3k+\ell$ or $i > k$ . On the other hand, the middle two positions of $\underline{a}$ remain constant in both sections, so $i = n-2k-\ell$ and $2i+j = n-k-\ell$ . Subtraction gives $i+j = k$ or $i \leq k$ .

(1c) also breaks into cases depending on the segments. We assume $\underline{a} = C^r_{ij} = (C^{4n-r}_{k\ell})^*$ , with $0 < j < (n-3i)/2$ and $0 < \ell < (n-3k)/2$ . Here the arguments do not group together as cleanly. One element of such a conjugate pair occurs at least as high as the middle rank in one chain. Call this chain $C_{ij}$ . For ease of comparison, we have recorded $C_{ij}$ and $C^*_{k\ell}$ in Table 2. Since $3n-3i-3j < 2n$ , $\underline{a}$ lies in segment 4 , 5 , or 6 of $C_{ij}$ . Since $n+3k+2\ell < 2n$ , $\underline{a}$ lies in segment 3 , 4 , 5 , or 6 of $C^*_{k\ell}$ . Assume $\underline{a} \in ({}^pC_{ij} \cap {}^qC_{k\ell})$ .

We first notice $p = 4$ is impossible, as it would imply $\ell \leq 0$ . We handle the remaining cases individually. Again we equate corresponding positions in $\underline{a}$ . The requirements on j and $\ell$ figure prominently. For example, $i+j \leq k$ and $i \geq k+\ell$ give us a contradiction, as do $n-3i-j \leq \ell$ and $n-3k-\ell \leq j$ .

$p = 6$ , $q = 6$ . $a_2 \Rightarrow 2i+j = 2k+\ell$ . $a_3 \Rightarrow i \leq k$ . $a_1 \Rightarrow 3i+j \geq 3k+\ell$ . Subtracting $a_2$ implies $i \geq k$ . So $(i,j) = (k,\ell)$ , and this is the case where the top and bottom of the chain are conjugate.

$p = 5$ , $q = 5$ . $a_3 \Rightarrow i+j = k$ . $a_2 \Rightarrow 2i+j \geq 2k+\ell$ . Subtracting $a_3$ implies $i \geq k+\ell$ .

$p = 6$ , $q = 5$ . $a_3 \Rightarrow k \geq i$ . $a_1 \Rightarrow n-3i-j = \ell$ . Substituting for i gives $n-3k-\ell \leq j$ . As mentioned earlier, this is a contradiction since both $3i+2j$ and $3k+2\ell$ must be less than n .

| rank | $C_{ij}$ | segment | $C^*_{k\ell}$ | rank |
|---|---|---|---|---|
| 4n-6i-2j | (n-3i-j , n-2 i-j , n-i , n) | | (n-3k-ℓ , n-2k-ℓ , n-k , n) | 4n-6k-2ℓ |
| | ⋮ | 6 | ⋮ | |
| 4n-68-3j | (n-3i-j , n-2 i-j , n-i-j , n) | | (ℓ , n-2k-ℓ , n-k , n) | 3n-3k |
| | ⋮ | 5 | ⋮ | |
| 3n-3i-2j | (0 , n-2i-j , n-i-j , n) | | (ℓ , k+ℓ , n-k , n) | 2n+2ℓ |
| | ⋮ | 4 | ⋮ | |
| 3n-3i-3j | (0 , n-2i-j , n-i-j , n-j) | | (ℓ , k+ℓ , 2k+ℓ , n) | n+3k+3ℓ |
| | ⋮ | 3 | ⋮ | |
| 2n-2j | (0 , i , n-i-j , n-j) | | (0 , k+ℓ , 2k+ℓ , n) | n+3k+2ℓ |
| | ⋮ | 2 | ⋮ | |
| n+3i | (0 , i , 2i+j , n-j) | | (0 , k+ℓ , 2k+ℓ , 3k+ℓ) | 6k+3ℓ |
| | ⋮ | 1 | ⋮ | |
| 6i+2j | (0 , i , σi+j , 3i+j) | | (0 , k , 2k+ℓ , 3k+ℓ) | 6k+2ℓ |

Table 2.

$p = 5$ , $q = 6$ . $a_3 \Rightarrow i+j = k$ . $a_2 \Rightarrow 2i+j = 2k+\ell$ . Subtracting $a_3$ implies $i = k+\ell$ , so $j = \ell = 0$ .

$p = 6$, $q = 4$. $a_1 \Rightarrow n-3i-j = \ell$ . $a_2 \Rightarrow n-2i-j = k+\ell$ . Subtracting $a_1$ gives $i = k$ . Substituting in $a_2$ yields $n-3k-\ell = j$ , giving the same contradiction as in $(p,q) = (6,5)$ .

$p = 5$, $q = 4$. $a_1 \Rightarrow n-3i-j > \ell$ (equality returns us to the previous case). $a_2 \Rightarrow n-2i-j = k+\ell$ . Subtracting $a_1$ gives $i < k$ . $a_3 \Rightarrow n-i-j \geq 2k+\ell$ . Subtracting $a_2$ gives $i \geq k$ .

$p = 6$ , $q = 3$ . Lest $p-q$ be smaller, the requirement on ranks is $4n-3i-3j < n+3k+3\ell$ , so $n-2i-j < k+\ell$ . But $a_2 \Rightarrow n-2i-j = k+\ell$ .

$p = 5$ , $q = 3$. $a_2 \Rightarrow n-2i-j = k+\ell$ . $a_3 \Rightarrow n-i-j = 2k+\ell$ . Subtracting $a_1$ yields $i = k$ . Substituting this in the two previous equations gives the familiar contradiction $n-3i-j = \ell$ and $n-3k-\ell = j$ .

This completes the proof of (1).

Details of Step 2. We begin with (2a). The top element of segment 4 in $C_{ij}$ has rank $3n-3i-2j \geq 2n$ , so every $C_{ij}$ has a 0 in the first position of its middle rank element. The bottom rank of segment 3 in $D_{ij}$ is $n+3i+2j+2 < 2n-1$ , so $D_{ij}$ has a positive first position in its middle rank element. The non-decreasing sequences of length 4 which start with 0 , end in k , and sum to 2n run from $(0, 2n-2k, k, k)$ to $(0, \lfloor(2n-k)/2\rfloor, \lceil(2n-k)/2\rceil, k)$ when $n \geq k \geq \lceil 2n/3 \rceil$ . So, we want the number of $C_{ij}$'s to be $\sum_{\lceil 2n/3\rceil \leq k \leq n} k - \lceil(2n-k)/2\rceil + 1$ . Similarly, the elements covered by $D_{ij}$'s run from $(k, k, n-2k, n)$ to

$(k, \lfloor (n-k)/2 \rfloor, \lceil (n-k)/2 \rceil, n)$ for $1 \le k \le \lfloor n/3 \rfloor$, for a total of

$$\sum_{1 \le k \le \lfloor n/3 \rfloor} \lfloor (n-k)/2 \rfloor - k + 1 .$$

On the other hand, the number of solutions to $3i+2j \le n$ is

$\sum_{0 \le i \le \lfloor n/3 \rfloor} 1 + \lfloor (n-3i)/2 \rfloor$ and to $3i+2j \le n-3$ is

$\sum_{0 \le i \le \lfloor n/3 \rfloor -1} 1 + \lfloor (n-3i-3)/2 \rfloor$. These turn into the desired summations when $i$ is set to $n-k$ in the first case and $k-1$ in the second.

We wish to combine the summations. Separating the $i = 0$ term from the first and adjusting the index in the second, the total number $f(n)$ of chains becomes

$$f(n) = 1 + \lfloor n/2 \rfloor + 2 \sum_{1 \le i \le \lfloor n/3 \rfloor} (1 + \lfloor (n-3i)/2 \rfloor) .$$

To compute the summation, we pair terms for consecutive values of $i$. If $\lfloor n/3 \rfloor$ is odd, we separate $i = \lfloor n/3 \rfloor$. Adding the terms for $i = 2k-1$ and $i = 2k$ gives $2 + \lfloor (n-6k+3)/2 \rfloor + \lfloor (n-6k)/2 \rfloor = n+3-6k$. There are $\lfloor n/6 \rfloor$ pairs altogether, and $\sum_{1 \le k \le \lfloor n/6 \rfloor} (n+3-6k) = (n+3)\lfloor n/6 \rfloor - 3\lfloor n/6 \rfloor \lfloor (n+6)/6 \rfloor$. When $\lfloor n/3 \rfloor$ is odd, the term $1 + \lfloor (n-3\lfloor n/3 \rfloor)/2 \rfloor$ remains. This is 1 if $n \equiv 3, 4 \bmod 6$, but 2 if $n \equiv 5 \bmod 6$.

Summarizing, if $n \equiv r \bmod 6$, $0 \le r \le 5$, then the total number of chains is

$$f(n) = \lfloor n/2 \rfloor + 2(n+3)\lfloor n/6 \rfloor - 6\lfloor n/6 \rfloor \lfloor (n+6)/6 \rfloor + \begin{cases} 1 & ; r = 0, 1, 2 \\ 3 & ; r = 3, 4 \\ 5 & ; r = 5 \end{cases}$$

$$= \lfloor n/2 \rfloor + (n+3)(n-r)/3 - (n-r)(n-r+6)/6 + \begin{cases} 1 & ; r = 0, 1, 2 \\ 3 & ; r = 3, 4 \\ 5 & ; r = 5 \end{cases}$$

Next we consider (2b). If n is even, a new chain $C_{ij}$ occurs for even values of i with $0 \leq i \leq \lfloor n/3 \rfloor$, and a new $D_{ij}$ for odd values of i with $1 \leq i \leq \lfloor n/3 \rfloor -1$. Similarly, when n is odd we have a new $D_{ij}$ for even i with $1 \leq i \leq \lfloor n/3 \rfloor -1$ and a new $C_{ij}$ for odd i with $1 \leq i \leq \lfloor n/3 \rfloor$.

To sum the number of elements in these chains, we can again pair consecutive terms. For the total number g(n) of these elements, we have

$$g(n) = \begin{cases} |C_{0,n/2}| + \sum\limits_{1 \leq k \leq \lfloor n/6 \rfloor} |D_{2k-1,(n-6k)/2}| + |C_{2k,(n-6k)/2}| & ;\ n \text{ even} \\ \sum\limits_{0 \leq k \leq \lfloor (n-3)/6 \rfloor} |D_{2k,(n-6k-3)/2}| + |C_{2k+1,(n-6k-3)/2}| & ;\ n \text{ odd} \end{cases}$$

Since $|C_{ij}| = 4(n-3i-j)+1$ and $|D_{ij}| = 4(n-3i-j)-5$, this quickly becomes

$$g(n) = \begin{cases} 1 + 2n + \sum\limits_{1 \leq k \leq \lfloor n/6 \rfloor} 4(n-6k) + 8 & ;\ n \text{ even} \\ \sum\limits_{0 \leq k \leq \lfloor (n-3)/6 \rfloor} 4(n-6k)-4 & ;\ n \text{ odd} \end{cases}$$

$$= \begin{cases} 1 + 2n + 4(n+2) \lfloor n/6 \rfloor - 12 \lfloor n/6 \rfloor \lfloor (n+6)/6 \rfloor & ;\ n \text{ even} \\ 4(n-1) \lfloor (n-3)/6 \rfloor - 12 \lfloor (n-3)/6 \rfloor \lfloor (n+3)/6 \rfloor & ;\ n \text{ odd} \end{cases}$$

$$= \begin{cases} 1 + 2n + 2(n+2)(n-r)/3 - (n-r)(n-r+6)/3 & ;\ r = 0,2,4 \\ 2(n-1)(n-r+6)/3 - (n-r)(n-r+6)/3 & ;\ r = 315 \\ 2(n-1)^2/3 - (n-7)(n-1)/3 & ;\ r = 1 \end{cases}$$

For (2c), we need only compute $4f(n-1) + g(n)$ , which b&comes simple algebraic manipulation when we consider a particular congruence class of n modulo 6 . Beginning with $r = 1$ , we easily obtain expressions like

$r = 1$: $4 + (n-1)(n+3)$

$r = 2$: $2n + 5 + (n-2)(n+2)$

$r = 3$: $4 + (n-1)(n+3)$

$r = 4$: $4n + 9 + (n+2)(n-4)$

$r = 5$: $2n + 10 + (n-5)(n+5)/3 + 2(n-1)(n+1)/3$

$r = 0$: $4n + 17 + 2(n-6)(n+4)/3 + n(n-2)/3$

all of which reduce to $(n+1)^2$ .

This completes the proof.

## References


[1] J. R. Griggs, "Sufficient conditions for a symmetric chain order," SIAM J. Applied Math. 32 (1977), 807-809.

[2] B. Lindström, "A partition of L(3, n) into saturated symmetric chains" (preprint).

[3] R. P. Stanley, "An application of algebraic geometry to extremal set theory" (preprint).