NOTE ON THE PRACTICAL SIGNIFICANCE OF THE DRAZIN INVERSE

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Abstract

The solution of the differential system $Bx = Ax + f$ where A and B are $n \times n$ matrices and $A - \lambda B$ is not a singular pencil may be expressed in terms of the Drazin inverse. It is shown that there is a simple reduced form for the pencil $A - \lambda B$ which is adequate for the determination of the general sobution and that although the Drazin inverse could be determined efficiently from this reduced form it is inadvisable to do so.

1 INTRODUCTION

In a recent paper [2] the solution of the differential system

$$B\dot{x} = Ax + f(t), \tag{1.1}$$

where B and A are $n \times n$ matrices and f is an n-vector has been discussed in terms of the Drazin inverse. Although this work gives considerable insight into the nature of the general solution of (1.1) it should not be assumed that because the explicit solution can be expressed directly in terms of the Drazin inverse that economical algorithms will involve its explicit computation.

Numerical analysts will be familiar with this in connexion with the simpler problem $Ax = b$ where A is non-singular. Although the solution is given by $x = A^{-1}b$ it is seldom advisable to compute the inverse explicitly. However algorithms for solving $Ax = b$ based on direct methods do provide the basic tools for the efficient computation of A^{-1} if that should be required; we might therefore expect that practical algorithms for solving (1.1), or closely related algorithms, would provide effective methods for computing the Drazin inverse and this is indeed true.

2 THE DRAZIN INVERSE

If A is an $n \times n$ matrix then the Drazin inverse [4] of A is the matrix X satisfying the relations

$$\begin{aligned} &\text{(i)} \quad AX = XA \\ &\text{(ii)} \quad XAX = X \\ &\text{(iii)} \quad XA^{k+1} = A^{k}, \text{ where } k = \mathrm{Ind}(A). \end{aligned}$$

$\mathrm{Ind}(A)$, the index of A, is the smallest non-negative integer for which $\mathrm{rank}\,(A^{k}) = \mathrm{rank}\,(A^{k+1})$.

The existence and uniqueness of A may be proved as follows. The proof is given in matrix terms since we shall need to work in these terms in subsequent sections. Let J be the Jordan canonical form of A, and suppose J is expressed as the direct sum of C and N where C is associated with the non-zero eigenvalues and N is associated with the zero eigenvalues and is therefore nil-potent. We may write

$$A = T \left[\begin{array}{c|c} C & 0 \\ \hline 0 & N \end{array}\right] T^{-1}, \tag{2.1}$$

where C is non-singular and N is nil-potent. If k is the smallest integer for which $N^k = 0$ it is clear that k is the index of A since

$$A^k = T \left[\begin{array}{c|c} C^k & 0 \\ \hline 0 & 0 \end{array}\right] T^{-1}, \quad A^{k+1} = T \left[\begin{array}{c|c} C^{k+1} & 0 \\ \hline 0 & 0 \end{array}\right] T^{-1} \tag{2.2}$$

and rank (A^k) = rank (A^{k+1}) = order of C. On the other hand rank $(A^p) >$ rank (A^{p+1}) when $p < k$. Obviously k is the dimension of the largest Jordan submatrix associated with a zero eigenvalue.

Any $n \times n$ matrix X may be expressed in the form $X = TYT^{-1}$ and relations (i), (ii) and (iii) are satisfied if and only if

(iv) $JY = YJ$

(v) $YJY = Y$

(vi) $YJ^{k+1} = J^k$

where

$$J = \left[\begin{array}{c|c} C & 0 \\ \hline 0 & N \end{array}\right]. \tag{2.3}$$

Partitioning Y conformally with J we may write

$$Y = \left[\begin{array}{c|c} P & Q \\ \hline R & S \end{array}\right]. \tag{2.4}$$

Equation (iv) then gives

$$\left.\begin{array}{l} CP = PC \text{ (a)}, \quad CQ = QN \text{ (b)} \\ NR = RC \text{ (c)}, \quad NS = SN \text{ (d)} \end{array}\right\}. \tag{2.5}$$

From (b) we have

$$CQN^{k-1} = QN^{k} = 0 \quad . \tag{2.6}$$

Hence $QN^{k-1} = 0$ since C is non-singular. Continuing in this way we have successively $QN^{k-2} = 0$, $QN^{k-3} = 0$, ..., $Q = 0$. Similarly from (c) $R = 0$. Now from (v) and (d)

$$SNS = S \text{ and } S^{2}N = S \ . \tag{2.7}$$

Hence

$$S^{2}N^{k} = SN^{k-1} \text{ giving } SN^{k-1} = 0 \ . \tag{2.8}$$

Continuing in this way $SN^{k-2} = 0$, SN^{k-3}, . ..) $s = 0$. Finally from (vi)

$$PC^{k+1} = C^{k} \text{ giving } P = C^{-1} \tag{2.9}$$

and hence

$$X = T \left|\begin{array}{c|c} C^{-1} & 0 \\ \hline 0 & 0 \end{array}\right| T^{-1} \ , \tag{2.10}$$

showing that X is uniquely determined. In proving this result we did not make use of the fact that C and N were the direct sum of Jordan matrices but merely that they were non-singular and nil-potent respectively. Hence to derive the Drazin inverse it is not necessary to obtain the Jordan canonical form itself but merely the identification of the nil-potent part, a much simpler objective.

When A is non-singular X is obviously A^{-1}, the usual inverse. Notice that it is not generally true that $AXA = A$ and hence a solution of a compatible system $AX = b$ is not, in general, given by $x = Xb$.

3 COMPUTATION OF THE DRAZIN INVERSE

We have shown that the Drazin inverse of A is available if we have expressed A in the form

$$A = T \left[\begin{array}{c|c} C & 0 \\ \hline 0 & N \end{array}\right] T^{-1} \quad , \tag{3.1}$$

where C is non-singular and N is nil-potent. A factorization of that form in which T is unitary has in fact been derived by Golub and Wilkinson [6]. In that factorization the singular value decomposition was used so as to give the maximum numerical stability. A similar reduction could be achieved by a whole range of elementary transformations and this we now describe in general terms.

We denote the original matrix by $A^{(1)}$. In the rth step a similarity transformation, based on multiplications with elementary matrices is applied to $A^{(r)}$ to give $A^{(r+1)}$. The general form of the matrices $A^{(r)}$ is adequately illustrated by the fact that

$$A^{(4)} = \left[\begin{array}{c|c|c|c} A^{(4)}_{44} & A^{(4)}_{43} & A^{(4)}_{42} & A^{(4)}_{41} \\ \hline 0 & 0 & A^{(4)}_{32} & A^{(4)}_{31} \\ \hline 0 & I\ 0 & 0 & A^{(4)}_{21} \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \} n_3 \\ \} n_2 \\ \} n_1 \end{array} \quad , \tag{3.2}$$

$\underbrace{\quad}_{n_3}\ \underbrace{\quad}_{n_2}\ \underbrace{\quad}_{n_1}$

where the significance of the n_i will become apparent in the description of the rth step which is as follows.

If the-matrix $A^{(r)}_{rr}$ is non-singular the reduction is complete. Otherwise premultiply $A^{(r)}_{rr}$ with a sequence of elementary transformations, the product of which is denoted by $Q^{(r)}$, such that

$$Q^{(r)} A^{(r)}_{rr} = \left[\begin{array}{c} B^{(r)} \\ \hline 0 \end{array}\right] \} n_r \quad , \tag{3.3}$$

where n_r is the nullity of $A^{(r)}_{rr}$. The matrices involved in $Q^{(r)}$ may be unitary

(orthogonal, if real) or may be elementary matrices corresponding to elimination techniques. If $A^{(1)}$ had small integer elements the use of rational numbers enables this reduction to be done exactly. Note that $B^{(r)}$ need not be trapezoidal so that this reduction can be achieved entirely by pre-multiplications. If we now post-multiply by $(Q^{(r)})^{-1}$ we may write

$$Q^{(r)} A^{(r)}_{rr} (Q^{(r)})^{-1} = \left[\begin{array}{c|c} A^{(r+1)}_{r+1,r+1} & A^{(r+1)}_{r+1,r} \\ \hline 0 & 0 \end{array}\right]_1 . \qquad (3.4)$$

Writing

$$T^{(r)} = \left[\begin{array}{c|c} Q^{(r)} & 0 \\ \hline \mathbf{0} & I \end{array}\right] \qquad (3.5)$$

where $T^{(r)}$ is of order n, then $A^{(r+1)} = T^{(r)} A^{(r)} (T^{(r)})^{-1}$ is again of the required form. Notice that the pre-multication with $T^{(r)}$ affects only the leading block row of $A^{(r)}$, while the post-multiplication affects only the principal leading submatrix. We must have $n_r \leq n_{r-1}$ since if $n_r > n_{r-1}$, this would imply that in the preceding stage n_{r-1} was not the full nullity. Indeed the $A^{(k)}_{i+1,i}$ must be of full row rank at every stage for the same reason. If the matrix $A^{(1)}$ is entirely nil-potent then we must reach an $A^{(k)}_{kk}$ which is null and the final matrix is of the block form illustrated by

$$\begin{bmatrix} 0 & X & X & X \\ 0 & 0 & X & X \\ & & & \\ 0 & 0 & 0 & X \end{bmatrix} . \qquad (3.6)$$

Otherwise we terminate with an $A^{(k+1)}_{k+1,k+1}$ which is non-singular. (In using the symbol k we are anticipating that this is the index of $A^{(1)}$). In this second case we can annihilate all blocks in the first row except $A^{(k+1)}_{k+1,k+1}$ by further similarity transformations. This is adequately illustrated by the case when $k = 3$ for which $A^{(4)}$ is as in (3.2) with $A^{(4)}_{44}$ non-singular. Post-multiplication with

$$P_3 = \begin{bmatrix} I & -(A_{44}^{(4)})^{-1}A_{43}^{(4)} & & \\ 0 & I & & \\ & & I & \\ & & & I \end{bmatrix} \qquad (3.7)$$

annihilates $A_{43}^{(4)}$ and leaves all other submatrices unaltered. Pre-multiplication with $P_3^{(-1)}$ preserves all the null matrices and changes $A_{42}^{(4)}$ and $A_{41}^{(1)}$. The (4,2) and (4,1) blocks may be annihilated successively in a similar way.

Thus according as $A^{(1)}$ is entirely nil-potent or not we achieve a reduction to one or other of the forms illustrated by

$$\begin{bmatrix} 0 & X & X & X \\ 0 & 0 & X & X \\ & & & \\ 00 & 00 & 00 & X \end{bmatrix} \text{ or } \begin{bmatrix} C & 0 & 0 & 0 \\ 0 & 0 & X & X \\ & & & \\ )0 & 00 & 00 & X0 \end{bmatrix} \qquad (3.8)$$

with C non-singular. We may denote this final matrix by

$$N \text{ or } \begin{bmatrix} C & \\ & N \end{bmatrix} \qquad (3.9)$$

in the two cases. Obviously $N^k = 0$ while it is easy to see that since the (i,i+1) blocks are all of full row rank $N^{\ell} \neq 0$ ($\ell < k$). Hence k is indeed the index.

The Drazin inverse could now be computed explicity using the product of all the transformation matrices but it would usually be more expedient to keep it in factorized form.

4 THE SOLUTION OF THE DIFFERENTIAL SYSTEM

When B is non-singular the system (1.1) may be written in the form

$$\dot{x} = B^{-1}Ax + B^{-1}f \ . \qquad (4.1)$$

There is a solution corresponding to any f and for arbitrary initial values x_0. This solution may be expressed in terms of $\exp(B^{-1}At)$. Singularity of A in no

way affects the explicit form of the solution. Although this is a non-trivial matter we shall assume, in common with the paper we have referred to, that we have satisfactory algorithms for it.

When B is singular but A is non-singular (1.1) may be written in the form

$$A^{-1}B\dot{x} = x + A^{-1}f \tag{4.2}$$

ie $$K\dot{x} = x + g \text{ (say)} . \tag{4.3}$$

The existence and nature of the solution may be examined in terms of the Drazin inverse of K but there seems to be little point in computing the latter explicitly, Indeed if

$$K = T^{-1}\begin{bmatrix} C & \\ & N \end{bmatrix} T, \tag{4.4}$$

then

$$\begin{bmatrix} C & \\ & N \end{bmatrix} T\dot{x} = Tx + Tg , \tag{4.5}$$

or

$$\begin{bmatrix} C & \\ & N \end{bmatrix} \begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} p \\ q \end{bmatrix} , \tag{4.6}$$

$$\begin{bmatrix} y \\ z \end{bmatrix} = Tx , \quad \begin{bmatrix} p \\ q \end{bmatrix} = Tg . \tag{4.7}$$

Hence

$$C\dot{y} = y + p \tag{4.8}$$

$$N\dot{i} = z + q . \tag{4.9}$$

Since $N^k = 0$, (4.9) gives

$$0 = N^{k-1}z + N^{k-1}q . \tag{4.10}$$

Multiplying (4.9) by N^{k-2} and substituting from (4.10)

$$-N^{k-1}\dot{q} = N^{k-2}z + N^{k-2}q \tag{4.11}$$

and continuing in this way

$$z = -\left[I + ND + \ldots + N^{k-1}D^{k-1}\right] q \text{ where } D = \frac{d}{dt}. \tag{4.12}$$

Notice that we must have

$$z_0 = \left(-\left[I + ND + \ldots + N^{k-1}D^{k-1}\right] q\right)_0 \tag{4.13}$$

and since the components of z_0 are linear combinations of those of x_0 this means that the initial x_0 must satisfy certain conditions for a solution to be possible. Provided these consistency conditions are satisfied there is then a unique solution corresponding to any q, assuming that it has k-l derivatives. We observe that in the homogeneous case q = 0, and the only solution of (4.9) is z = 0.

Since C is non-singular the system (4.8) has a unique solution corresponding to any initial y_0 and this may be expressed in terms of $\exp(C^{-1}t)$.

The solution described above has been given in the spirit of the work based on the use of the Drazin inverse, but we would submit that even here too much attention has been paid to obtaining explicit expressions. It is more economical to-work with the form exemplified in (3.2). We describe this below and for convenience of presentation we assume that k = 3 and omit upper suffices. A transformation of variables has then reduced the original system to one of the form

$$\begin{bmatrix} A_{44} & A_{43} & A_{42} & A_{41} \\ & & & \\ & & & \\ \vdots & \vdots & A_{32}\,0\,0 & A_{31}\,0\,1 \end{bmatrix} \begin{bmatrix} \dot{y}_4 \\ \dot{y}_3 \\ \dot{y}_2 \\ \dot{y}_1 \end{bmatrix} = \begin{bmatrix} y_4 \\ y_3 \\ y_2 \\ y_1 \end{bmatrix} + \begin{bmatrix} g_4 \\ g_3 \\ g_2 \\ g_1 \end{bmatrix} \tag{4.14}$$

where the blocks on the diagonal are square and A_{44} is non-singular. The matrix

$$\begin{bmatrix} 0 & A_{32} & A_{31} \\ 0 & 0 & A_{21} \\ 0 & 0 & 0 \end{bmatrix} \tag{4.15}$$

is the N and A_{44} is the C of our previous analysis.

The relation (4.14) gives successively

$$y_1 = -g_1,\ y_2 = -g_2 - A_{21}\dot{y}_1,\ y_3 = -g_3 - A_{31}\dot{y}_1 - A_{32}\dot{y}_2 \tag{4.16}$$

Finally we have

$$A_{44}\dot{y}_4 = y_4 + (g_4 - A_{41}\dot{y}_1 - A_{42}\dot{y}_2 - A_{43}\dot{y}_3) \tag{4.17}$$

and at this stage y_1, y_2 and y_3 and hence $\dot{y}_1$, $\dot{y}_2$ and $\dot{y}_3$ have already been determined. Notice that when we describe the solution in these terms there is no need to annihilate the blocks A_{43}, A_{42} and A_{41} as we did in section 3 when describing a reduction to the form

$$\begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix} . \tag{4.18}$$

Now we merely have terms involving these A_{4i} on the right of (4.17). At the end of the next section we show how the volume of work may be reduced even further.

5 SINGULAR A AND B

When both A and B are singular one cannot proceed as in the previous section. The use of the Drazin inverse has been concerned with the case when $\det(A-\lambda B) \not\equiv 0$ ie when the pencil $A-\lambda B$ is non-singular in the Kronecker sense (see eg [3, 5, 8]). The matrix $A-cB$ is then non-singular for any c which is not a root of the equation $\det(A-\lambda B) = 0$. If one takes any such c then the system (1.1) is equivalent to

$$(A-cB)^{-1}B\dot{x} = (A-cB)^{-1}Ax + (A-cB)^{-1}f \qquad (5.1)$$

or

$$\hat{B}\dot{x} = \hat{A}x + \hat{f} \ . \qquad (5.2)$$

It may be readily verified that $\hat{B}\hat{A} = \hat{A}\hat{B}$. The explicit solution of (5.2) may be expressed in terms of the Drazin inverse of $\hat{B}$. Although, of course, the derived solution must be independent of c, its introduction is undesirable. In practice it would be important for A-cB to be, not merely non-singular, but well conditioned with respect to inversion, otherwise there will be a loss of accuracy which may be far greater than that resulting from the inherent sensitivity of the problem.

It will be appreciated that one will not necessarily know in advance whether A and B are singular or indeed whether $\det(A-\lambda B) \equiv 0$. The method described below, which is analogous to that described in section 3 for the computation of the Drazin inverse of a matrix, does not require any previous knowledge and does not require the use of the arbitrary scalar c.

We observe that if P and Q are non-singular then pre-multiplication of the system (1.1) with P and the transformation $x = Qy$ transforms it to the equivalent system

$$PBQ\dot{y} = PAQy + Pf \ . \qquad (5.3)$$

In our algorithm P and Q are determined as products of elementary matrices in such a way that (5.3) is typically of the form illustrated by

$$\begin{bmatrix} B_{44}^{(4)} & B_{43}^{(4)} & B_{42}^{(4)} & B_{41}^{(4)} \\ 0 & 0 & B_{32}^{(4)} & B_{31}^{(4)} \\ 0 & 0 & 0 & B_{21}^{(4)} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_4 \\ \dot{y}_3 \\ \dot{y}_2 \\ \dot{y}_1 \end{bmatrix} = \begin{bmatrix} A_{44}^{(4)} & A_{43}^{(4)} & A_{42}^{(4)} & A_{41}^{(4)} \\ 0 & A_{33}^{(4)} & A_{32}^{(4)} & A_{31}^{(4)} \\ 0 & 0 & A_{22}^{(4)} & A_{21}^{(4)} \\ 0 & 0 & 0 & A_{11}^{(4)} \end{bmatrix} \begin{bmatrix} y_4 \\ y_3 \\ y_2 \\ y_1 \end{bmatrix} + \begin{bmatrix} g_4 \\ g_3 \\ g_2 \\ g_1 \end{bmatrix} \qquad (5.4)$$

The diagonal blocks are square and $A_{11}^{(4)}$, $A_{22}^{(4)}$, $A_{33}^{(4)}$ and $B_{44}^{(4)}$ are non-singular.

The matrices $B_{21}^{(4)}$ and $B_{32}^{(4)}$ are of full row rank. In general there are k steps, the process coming to an end when $B_{k+1,k+1}^{(k+1)}$ is non-singular.

Suppose we have performed r-l steps and $B_{r,r}^{(r)}$ is still singular. In this case $B_{r,r}^{(r)}$ may be reduced to the form

$$\left[\begin{array}{c} E^{(r)} \\ \hline 0 \end{array}\right] \Big\} n_r \qquad (5.5)$$

by pre-multiplication with elementary matrices. Here n_r is the nullity of $B_{r,r}^{(r)}$ and $E^{(r)}$ is not required to be of upper trapezoidal form. If the same operations are performed on $A_{rr}^{(r)}$ the resulting matrix may be denoted by

$$\left[\begin{array}{c} F^{(r)} \\ \hline G^{(r)} \end{array}\right] \Big\} n_r \,. \qquad (5.6)$$

Now $G^{(r)}$ must be of full row rank n_r, since otherwise $A_{rr}^{(r)}$ and $B_{rr}^{(r)}$ share a common left-hand null vector and this would imply that $\det (A_{rr}^{(r)} - \lambda B_{rr}^{(r)}) \equiv 0$. Hence $G^{(r)}$ may be multiplied on the right by elementary matrices to give

$$\left[\, 0 \mid A_{rr}^{(r+1)} \,\right] \,, \qquad (5.7)$$

where $A_{rr}^{(r+1)}$ is non-singular. If these right-hand transformations are applied to the full matrices

$$\left[\begin{array}{c} E^{(r)} \\ \hline 0 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c} F^{(r)} \\ \hline G^{(r)} \end{array}\right] \qquad (5.8)$$

the resulting matrices may be denoted by

$$\left[\begin{array}{c|c} B_{r+1,r+1}^{(r+1)} & B_{r+1,r}^{(r+1)} \\ \hline 0 & 0 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c|c} A_{r+1,r+1}^{(r+1)} & A_{r+1,r}^{(r+1)} \\ \hline 0 & A_{rr}^{(r+1)} \end{array}\right] \,. \qquad (5.9)$$

The rth step is completely determined by the matrices $B_{rr}^{(r)}$ and $A_{rr}^{(r)}$ but if we apply the transformations to the full n × n matrices and to the current forcing vector we arrive at an rth derived system of the same form as the (r-1)th system from which we started. The $B_{r,r-1}^{(r+1)}$ must be of full row rank otherwise the n_{r-1} determined in the previous stage would have been incorrect.

If $\det(A-\lambda B) \not\equiv 0$ we must either reach a $B_{rr}^{(r)}$ which is non-singular or one which is completely null with $A_{rr}^{(r)}$ non-singular. If however $\det(A-\lambda B) \equiv 0$ this would be detected by the algorithm since we would reach a stage at which the $G^{(r)}$ of (5.6) was not of full rank and this would reveal itself when performing the elementary operations on $G^{(r)}$.

For simplicity of presentation let us assume that the process terminates when k = 3 so that the final system is as given in (5.4). We suppress the upper suffix for convenience. The solution is then given by

$$\begin{aligned} A_{11}y_1 &= -g_1 \\ A_{22}y_2 &= -g_2 - A_{21}y_1 - B_{21}\dot{y}_1 \\ A_{33}y_3 &= -g_3 - A_{31}y_1 - A_{32}y_2 - B_{31}\dot{y}_1 - B_{32}\dot{y}_2 \end{aligned} \qquad (5.10)$$

so that the components of y_1, y_2 and y_3 are all uniquely determined and the initial values must satisfy equations (5.10) for consistency. Finally

$$B_{44}\dot{y}_4 = A_{44}y_4 + (A_{43}y_3 + A_{42}y_2 + A_{41}y_1 - B_{43}\dot{y}_3 - B_{42}\dot{y}_2 - B_{41}\dot{y}_1 + g_4) \qquad (5.11)$$

and the vector in parenthesis is already determined. Since B_{44} is non-singular this has a unique solution for arbitrary initial y_4 which may be expressed in terms of $\exp(B_{44}^{-1}A_{44}t)$.

The elementary transformations on $G^{(r)}$ would usually be carried out in such a way that $A_{rr}^{(r+1)}$ would be at least triangular (though possibly even diagonal) according to the method used. The computation of the vectors y_1, y_2, y_3 from relations(5.10) would therefore be particularly convenient. As we remarked above if at any stage $G^{(r)}$ is not of full rank this would be exposed automatically in the execution of the algorithm. (We assume here that the algorithm used to

reduce $G^{(r)}$ is stable enough to detect rank reliably!). This can happen only if $\det(A-\lambda B) \equiv 0$. This situation is not usually covered by the use of the Drazin inverse. When $G^{(r)}$ has a rank deficiency of p then p linear relations must hold between components of f for the differential equations to be compatible. This is discussed in detail in [8]. However the general situation may be illustrated by considering what happens when $G^{(1)}$ has a rank deficiency of p. This means that the original system is equivalent to a system of the form

$$\begin{matrix} n_1-p \\ p \end{matrix}\left\{\left[\begin{array}{c} K \\ \hline 0 \\ \hline 0 \end{array}\right]\right. \dot{y} = \begin{matrix} n_1-p \\ p \end{matrix}\left\{\left[\begin{array}{c} L \\ \hline M \\ \hline 0 \end{array}\right]\right. y + g \quad , \tag{5.12}$$

where M is of full rank, n_1-p. Hence the last p components of g must be zero for the equations to be compatible, and the components of g are linear combinations of the original components of f.

When both A and B are singular but $\det(A-\lambda B) \not\equiv 0$, then when we reach the terminating non-singular $B^{(r)}_{rr}$ the corresponding $A^{(r)}_{rr}$ must be singular. This follows because the earlier $A^{(i)}_{ii}$ were non-singular and if $A^{(r)}_{rr}$ were non-singular this would imply non-singularity of A.

We have remarked that the solution may be expressed in terms of the Drazin inverse of $(A-cB)^{-1}B$ and the form of the solution is determined by the index of $(A-cB)^{-1}B$. The k introduced above is in fact this index as we now show. Denoting the successive $n \times n$ matrices derived by the algorithm by $A^{(\ )}$ and $B^{(r)}$ respectively, $A^{(k+1)} - cB^{(k+1)}$ has as its diagonal blocks

$$A^{(k+1)}_{k+1,k+1} - cB^{(k+1)}_{k+1,k+1},\ A^{(k+1)}_{kk},\ \ldots,\ A^{(k+1)}_{2,2},\ A^{(k+1)}_{..} \quad . \tag{5.13}$$

The last k of these and $B^{(k+1)}_{k+1\,k+1}$ are non-singular by definition of the algorithm. The first is non-singular for any c for which $\det(A^{(k+1)}_{k+1\,k+1} - cB^{(k+1)}_{k+1,k+1}) \not\equiv 0$ ie for almost all c. Obviously

$$X = \left[\ A^{(k+1)} - cB^{(k+1)}\right]^{-1} B^{(k+1)}$$

is block upper-triangular and its diagonal blocks are

$$\left[A^{(k+1)}_{k+1,k+1} - cB^{(k+1)}_{k+1,k+1}\right]^{-1} B^{(k+1)}_{k+1,k+1}, \; 0, \; \ldots, \; 0, \; 0 \; . \tag{5.14}$$

Further $X_{i,i-1} = (A^{(k+1)}_{ii})^{-1} B^{(k+1)}_{i,i-1}$ and hence is of full row rank for $2 \leq i \leq k$ since this is true of the $B^{(k+1)}_{i,i-1}$. Hence the k of our algorithm is the index of $\left[A^{(k+1)} - cB^{(k+1)}\right]^{-1} B^{(k+1)}$ and since $A^{(k+1)} = PAQ$, $B^{(k+1)} = PBQ$ for some non-singular P and Q, our k is the index of $(A-cB)^{-1}B$.

The algorithm we have described works in terms of full $n \times n$ matrices at all stages in the reduction, though to be sure in later stages only parts of these matrices are affected by the transformation. We have presented the algorithm in this way in order to give a closer tie up with earlier work involving the Drazin inverse. However, if one were concerned with only one forcing vector f, or if indeed one were interested in several different forcing functions all of which were known at the time when the reduction was performed then a considerable economy would be achieved as follows. Suppose we have completed one stage of the reduction and have reached the reduced system

$$\begin{bmatrix} B^{(2)}_{22} & B^{(2)}_{21} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_2 \\ \dot{y}_1 \end{bmatrix} = \begin{bmatrix} A^{(2)}_{22} & A^{(2)}_{21} \\ 0 & A^{(2)}_{11} \end{bmatrix} \begin{bmatrix} y_2 \\ y_1 \end{bmatrix} + \begin{bmatrix} g_2 \\ g_1 \end{bmatrix} \; . \tag{5.15}$$

At this stage the variables in y_1 are completely determined and these variables undergo no further transformations. We have then

$$y_1 = -(A^{(2)}_{11})^{-1} g_1 \tag{5.16}$$

and

$$B^{(2)}_{22}\dot{y}_2 = A^{(2)}_{22}y_2 + \left\{g_2 - B^{(2)}_{21}\dot{y}_1 + A^{(2)}_{21}y_1\right\}_3$$

$$= A^{(2)}_{22}Y2 + f_2 \text{ (say)} \; . \tag{5.17}$$

Hence we can continue with a system of lower order. In this way we avoid

performing any transformations on $B_{21}^{(2)}$ and $A_{21}^{(2)}$ in the next step. The first stage is wholly typical; in the rth stage we determine n_r more variables and are left with a system in n_r fewer variables. Obviously if we are interested in the effect of several forcing functions we can deal with them all simultaneously. A similar reduction of effort may be achieved with the simpler algorithm of section 4.

6 NUMERICAL EXAMPLE

As an illustration of our algorithm we describe its performance on the example used by Campbell et al [2].

The system of differential equations is

$$A\dot{x} + Bx = b$$

$$\begin{bmatrix} -1 & 0 & 2 \\ 2 & 3 & 2 \\ 1 & 0 & -2 \end{bmatrix} \dot{x} + \begin{bmatrix} -27 & -22 & -17 \\ 18 & 14 & 10 \\ 0 & 1 & 2 \end{bmatrix} x = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \qquad (6.1)$$

where we have reordered the equations in order to avoid a row permutation during the course of the solution. This makes the process a little easier to follow. Naturally we have used rational elimination techniques. The authors gave the general solution to the homogeneous system as well as that corresponding to the forcing function b. For convenience of comparison we have followed the notation $A\dot{x} + Bx = b$ used by Campbell et al.

Exposing the row nullity of A gives

$$\begin{bmatrix} -1 & 0 & 2 \\ 2 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix} \dot{x} + \begin{bmatrix} -27 & -22 & -17 \\ 18 & 14 & 10 \\ -27 & -21 & -15 \end{bmatrix} x = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}. \qquad (6.2)$$

We now reduce the rows of B corresponding to the null rows of A. In fact there is only one such row and to facilitate comparison with Campbell et al we leave (3,1) as the non-zero element rather than (3,3). This involves the transformation

$$x = \left| \begin{bmatrix} 1 & -\frac{7}{9} & -\frac{5}{9} \\ & 1 & \\ & & 1 \end{bmatrix} Y \text{ or } \begin{matrix} y_1 = x_1 + \frac{7}{9}x + \frac{5}{9}x_3 \\ Y2 = x_2 \\ y_3 = x_3 \end{matrix} \right| \tag{6.3}$$

and leads to

$$\begin{bmatrix} -1 & \frac{7}{9} & \frac{23}{9} \\ 2 & \frac{13}{9} & \frac{8}{9} \\ 0 & 0 & 0 \end{bmatrix} \dot{y} + \begin{bmatrix} -27 & -1 & -2 \\ 18 & 0 & 0 \\ -27 & 0 & 0 \end{bmatrix} y = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} . \tag{6.4}$$

At this stage the singularity of B is exposed. The third equation gives

$$-27y_1 = 3 \quad \text{ie} \quad 9x1 + 7x_2 + 5x_3 + 1 = 0 \tag{6.5}$$

while for the homogeneous system

$$9x1 + 7x_2 + 5x_3 = 0 \tag{6.6}$$

Notice that these relations must hold for all values of t and therefore in particular for t = 0; at t = 0 they are in fact equations (35) and (29) respectively of Campbell et al.

Substituting $y_1 = -1/9$ into the first two equations and remembering that $Y2 = x_2$, $Y3 = x_3$ we have

$$\left.\begin{aligned} \frac{7}{9}\dot{x}_2 + \frac{23}{9}\dot{x}_3 - x_2 - 2x_3 &= -1 \\ \frac{13}{9}\dot{x}_2 + \frac{8}{9}\dot{x}_3 &= 2 \end{aligned}\right\} \tag{6.7}$$

and the solution is now trivial. The general solution is

$$\begin{aligned} x_1 &= -\frac{1}{18}(x_2(0) + 2x_3(0))e^{2/3t} - \frac{1}{18}(13x_2(0) + 8x_3(0)) - t - \frac{1}{9} \\ x_2 &= -\frac{1}{18}(8x_2(0) + 16x_3\ 0))e^{2/3t} + \frac{1}{18}(26x_2(0) + 16x_3(0)) + 2t \\ x_3 &= \frac{1}{18}(13x_2(0) + 26x_3(0))e^{2/3t} - \frac{1}{18}(13x_2(0) + 8x_3(0)) - t \end{aligned} \tag{6.8}$$

For the homogeneous case the general solution consists merely of the terms in (6.8) involving $x_2(0)$ and $x_3(0)$ with the others omitted. The solutions given here differ somewhat from those given by Campbell et al; this results from a trivial error made by them in the execution of their algorithm.

Of course this example is in some ways deceptively simple; however this is equally true of the solution obtained via the Drazin inverse. In general the system (6.7) above in which the matrix involving the derivatives is non-singular would be reached only after several stages of reduction (in fact k stages where k is the index associated with the relevant Drazin inverse). The solution of this reduced system can be expressed in terms of an exponential involving only an ordinary inverse.

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