ON GRID OPTIMIZATION FOR BOUNDARY VALUE PROBLEMS

by

R. Glowinski

STAN-CS-79-720
February 1979

COMPUTER SCIENCE DEPARTMENT
School of Humanities and Sciences
STANFORD UNIVERSITY

ON GRID OPTIMIZATION
FOR BOUNDARY VALUE PROBLEMS

*R. GLOWINSKI**

IRIA-LABORIA
Domaine de Voluceau, Rocquencourt
78150 Le Chesnay, France

and

Paris VI University
4, Place Jussieu, 75230 Paris cedex 05

ABSTRACT :

We discuss in this report the numerical procedures which can be used to obtain the optimal grid when solving by a finite element method a model boundary value problem of elliptic type modelling the potential flow of an incompressible inviscid fluid. Results of numerical experiments are presented.

* Funds for the support of this study have been allocated by the NASA-Ames Research Center, Moffet Field, California, under Interchange N° NCA2-OR745-816

Research supported in part by NSF Grant MCS78-11985.

CONTENTS

-o-

ACKNOWLEDGEMENTS

Most of the research in this report has been done during July and August 1978 while the author was visiting the Department of Computer Sciences of Stanford University and the Computational Fluid Dynamics Branch of NASA-Ames Research Center, under the Joint Research Interchange NCA2-OR745-816.

We would like to thank Professor G.H. GOLUB from Stanford University and Drs. T. HOLST and W.F.BALLHAUS from NASA Ames for their interest in this work, and Dr. A. MARROCCO from IRIA/LABORIA who coded the numerical experiments described in this report.

1. INTRODUCTION

Most boundary value problems of Mathematical Physics are solved by either finite difference or finite element methods ; both methods use a discretization mesh and for practical problems whose geometry is complicated, the grid corresponding to the mesh shows also a high degree of complication. A natural question which arises then is how to choose the discretization grid, for a given number of nodes, in order to minimize some functional of the error of approximation ; it is clear that a great deal of such functionals exist and the choice of one of them, in view of obtaining significant results is by itself a non trivial problem.

The Optimal grid problem is a complicated problem mainly for the two following reasons :

[1] It is a nonlinear problem even if the partial differential equation modelling the problem under consideration is linear. It means that the optimal grid is a function of the data producing a given solution.

[2] The exact solution is not known in general and the main difficulty in this Optimal grid problem is to find an error functional and a methodology of solution able to overcome this major difficulty (in view of some studies it is of course always possible to solve a problem with a very high accuracy using an highly refined - and therefore very costly - discretization grid, and then consider this solution as a reference solution, playing the role of the exact solution in the remaining part ot the study).

In this report we shall consider as a model problem the solution of a Poisson equation on a domain with a re-entrant corner. Such problems occur in Fluid Dynamics when considering the potential flows of incompressible inviscid fluids. Using a finite element approximation of this test problem we shall describe a numerical procedure to obtain the grid (or triangulation) which minimizes the truncation error

$$e_h = |u_h - u|_{1,\Omega} = \left(\int_\Omega |\nabla(u_h - u)|^2 dx\right)^{1/2}$$

where u (resp. u_h) is the exact (resp. approximate) solution. Numerical experiments will show how the mesh has to behave in the neighbourhood of the re-entrant corner if one wishes to minimize the above truncation error.

2. FORMULATION OF A MODEL PROBLEM.

Let Ω be a bounded domain of $\mathbb{R}^2$ whose boundary $\partial\Omega$ is denoted by Γ in the following. We suppose that $\Gamma = \Gamma_0 \cup \Gamma_1$, with $\Gamma_0 \cap \Gamma_1 = \emptyset$ (see Fig. 2.1 below) ; we shall suppose that $\int_{\Gamma_0} d\Gamma > 0$, where $d\Gamma$ is the superficial measure of Γ.

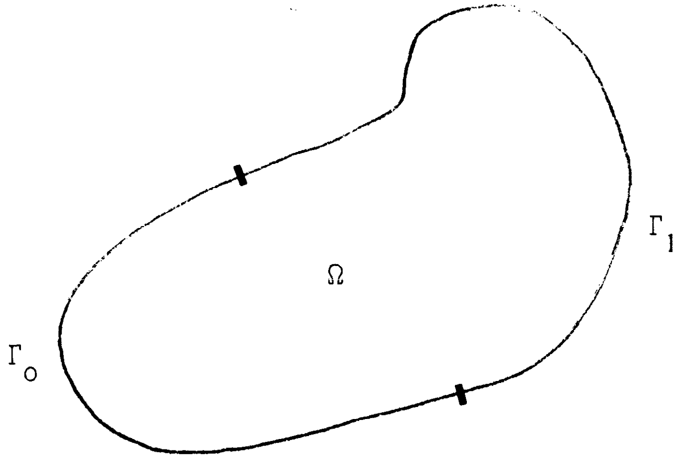


Figure 2.1

We consider on Ω the Poisson problem

$$(2.1)\quad \begin{cases} -\Delta\psi = f \text{ in } \Omega, \\ \psi|_{\Gamma_0} = g_0 \ , \ \dfrac{\partial\psi}{\partial n}\Big|_{\Gamma_1} = g_1 \ , \end{cases}$$

where f, g_0, g_1 are sufficiently smooth.

Let $x = \{x_1, x_2\}$ be the generic point of $\mathbb{R}^2$, we use the notation $dx = dx_1 dx_2$. Let introduce the (classical) space $H^1(\Omega)$ defined by

$$H^1(\Omega) = \{\phi | \phi, \frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2} \in L^2(\Omega)\}$$

and $V_0 \subset H^1(\Omega)$ defined by

$$V_0 = \{\phi | \phi \in H^1(\Omega) \ , \ \phi = 0 \text{ on } \Gamma_0\} \ .$$

Multiplying by ϕ the first equation in (2.1) and using Green's formula we obtain

$$\int_\Omega \nabla\psi \cdot \nabla\phi \, dx = \int_\Omega f\phi \, dx \qquad \forall \phi \in V_0 .$$

In fact it can be proved that (2.1) has a unique solution which is also the solution of the linear variational equation

$$(2.2)\quad \begin{cases} \underline{\text{Find}}\ \psi \in H^1(\Omega), \psi|_{\Gamma_0} = g_0\ ,\ \underline{\text{such that}}\ . \\ \int_\Omega \nabla\psi\cdot\nabla\phi\ dx = \int_\Omega f\phi\ dx + \int_{\Gamma_1} g_1\phi\ d\Gamma \quad \forall\phi\in V_0, \end{cases}$$

and conversely (see e.g. LIONS-MAGENES [1], NECAS [2], ODEN-REDDY [3] for such equivalence results).

The variational equation (2.2) is actually equivalent to the following problem from the Calculus of Variations

$$(2.3)\quad \begin{cases} \underline{\text{Find}}\ \psi \in H^1(\Omega),\ \psi|_{\Gamma_0} = g_0,\ \underline{\text{such that}} \\ J(\psi) \le J(\phi)\quad \forall\phi\in H^1(\Omega),\ \phi|_{\Gamma_0} = g_0 \end{cases}$$

where

$$J(\phi) = \frac{1}{2}\int_\Omega |\nabla\phi|^2 dx - \int_\Omega f\phi\ dx - \int_{\Gamma_1} g_1\phi\ d\Gamma\ .$$

Example : The problem below is a particular problem (2.1) with Ω as shown on Fig. 2.2 and-

$$(2.4)\quad \begin{cases} \Delta\psi = 0\ \text{in}\ \Omega, \\ \psi|_{\Gamma_0} = 0\ ,\ \frac{\partial\psi}{\partial n}|_{\Gamma_1}\ \underline{\text{as shown on}}\ \text{Fig. 2.2} \end{cases}$$

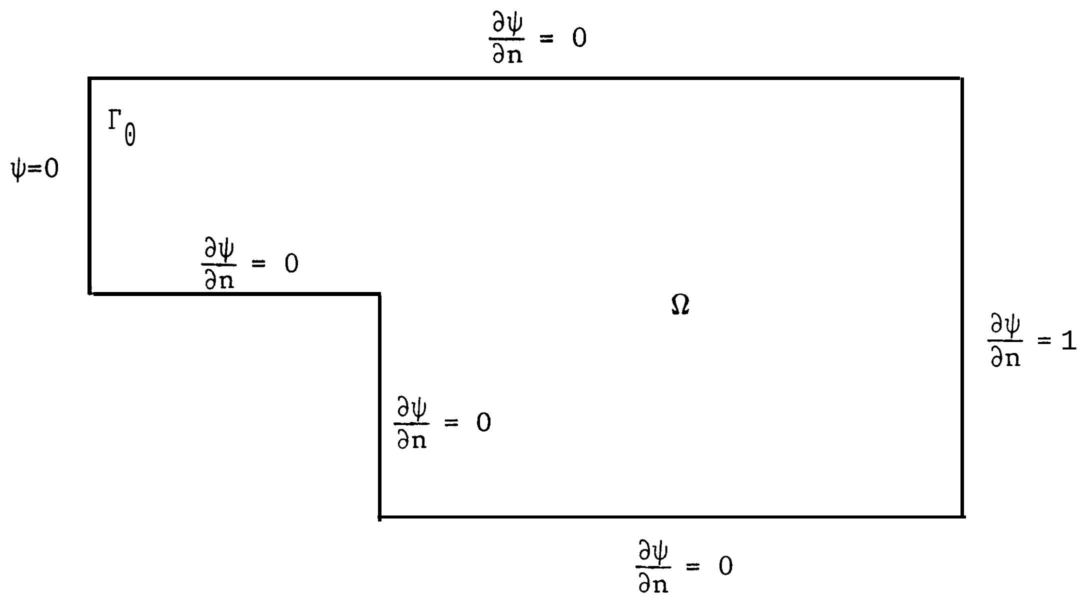


Figure 2.2

The Poisson problem (2.4) can be viewed as modelling the potential flow of an incompressible, inviscid fluid, in the cavity Ω ; the flow velocity $\vec{v}$ is given by $\vec{v} = \nabla\psi$. We shall give in Sec. 6 the results of numerical experiments concerning problem (2.4).

3. - FINITE ELEMENT APPROXIMATION OF THE MODEL PROBLEM.

3.1. Triangulation of Ω. Fundamental discrete spaces.

For simplicity we shall suppose that Ω is a bounded polygonal domain of $\mathbb{R}^2$ (as in the example of Sec. 2). To approximate (2.1) we shall use a finite element method. Let introduce a family $(\mathcal{T}_h)_h$ of triangulations of Ω obeying the following properties :

(i) $\mathcal{T}_h$ is a finite collection of triangles,

(ii) $\bigcup_{T \in \mathcal{T}_h} T = \overline{\Omega}$ ($\overline{\Omega}$: closure of R),

(iii) If $T, T' \in \mathcal{T}_h$ with $T \neq T'$ we only have the following possibilities

(a) $T \cap T' = \emptyset$,

(b) T, T' have a common vertex and only one,

(c) T, T' have a common side and only one.

As usual we denote by h the maximal side length in $\mathcal{T}_h$.
We define now from $\mathcal{T}_h$ an approximation V_h of $H^1(\Omega)$ by

$$V_h = \{\phi_h | \phi_h \in C^0(\overline{\Omega}),\ \phi_h|_T \in P_1\ \forall T \in \mathcal{T}_h\} ;$$

as usual

$$P_1 = \text{space of the polynomials in } x_1, x_2 \text{ of degree} \leq 1.$$

To approximate V_o we make the natural simplifying assumption

(3.1) The points of Γ at the interface of Γ_o and Γ_1 are vertices of $\mathcal{T}_h$.

We define then an approximation V_{oh} of V_o by

$$V_{oh} = \{\phi_h | \phi_h \in V_h\ ,\ \phi_h = 0 \text{ on } \Gamma_o\} .$$

The two spaces V_h and V_{oh} are finite dimensional spaces and

$$\dim(V_h) = \text{number of vertices in } \mathcal{T}_h,$$
$$\dim(V_{oh}) = \dim(V_h) - \left\{\begin{array}{l}\text{number of vertices on } \Gamma_o, \text{ including the nodes at} \\ \text{the interface of } \Gamma_o \text{ and } \Gamma_1\end{array}\right\}$$

From a computational point of view it is essential to have convenient vector basis for V_h and V_{oh} ; in this direction let us define

$$\Sigma_h = \{P | P \in \overline{\Omega},\ P \text{ vertex of } \mathcal{T}_h\} ,$$

$$\Sigma_{oh} = \{P | P \in \Sigma_h,\ P \notin \Gamma_o\}$$

(then $\dim(V_h) = \text{Card}(\Sigma_h)$, $\dim(V_{oh}) = \text{Card}(\Sigma_{oh})$).
To each $P \in \Sigma_h$ we associate a function w_P defined by

$$(3.2)\quad \begin{cases} w_P \in V_h, \\ w_P(P) = 1,\ w_P(Q) = 0 \quad \forall Q \in \Sigma_h,\ Q \neq P . \end{cases}$$

Then $\mathcal{B}_h = \{w_P\}_{P \in \Sigma_h}$ (resp. $\mathcal{B}_{oh} = \{w_P\}_{P \in \Sigma_{oh}}$) is a basis of V_h (resp. V_{oh}) and if $\phi_h \in V_h$ (resp. V_{oh}) we have the expansion

$$(3.3)\quad \phi_h = \sum_{P \in \Sigma_h} \phi_h(P) w_P$$

(resp.

$$(3.4)\quad \phi_h = \sum_{P \in \Sigma_{oh}} \phi_h(P) w_P) \quad .$$

3.2 The approximate problem.

We suppose in this sub-section that g_o is continuous, that f, g_1 are piecewise continuous and that their possible discontinuity lines or points are supported by sides or vertices of $\mathcal{T}_h$.

We define $V_{gh} \subset V_h$ by

$$V_{gh} = \{\phi_h | \phi_h \in V_h,\ \phi_h(P) = g_o(P)\ \forall P \in \Gamma_o \cap \Sigma_h\} .$$

To approximate (2.1) we approximate in fact the variational problem (2.2) by

$$(3.5)\quad \begin{cases} \text{Find } \psi_h \in V_{gh} \text{ such that} \\ \int_\Omega \nabla\psi_h \cdot \nabla\phi_h \, dx = \int_\Omega f_h \, \phi_h \, dx + \int_{\Gamma_1} g_{1h} \, \phi_h \, d\Gamma \ \ \forall \phi_h \in V_{oh} , \end{cases}$$

where f_h and g_{1h} are (for exemple) piecewise linear approximations of f and g_1. It can be proved (see e.g. [3], [4], [5]) that (3.5) has a unique solution, it can be also proved (see again [3]-[5]) that under reasonable assumptions on f, g_o, g_1 and the family $(\mathcal{T}_h)_h$ we have

$$(3.6)\quad \lim_{h\to 0} ||\psi_h - \psi||_{L^2(\Omega)} = 0 ,$$

$$(3.7)\quad \lim_{h\to 0} |\psi_h - \psi|_{1,\Omega} = 0$$

where, for $\phi \in H^1$, $|\phi|_{1,\Omega} = (\int_\Omega |\nabla\phi|^2 dx)^{1/2}$.

3.3. Formulation of the approximate problem as a linear system.

From a computational point of view it is more convenient to formulate (3.5) as a linear system. We observe that (3.5) is clearly equivalent to

$$(3.8)\quad \begin{cases} \text{Find } \psi_h \in V_{gh} \text{ such that} \\ \int_\Omega \nabla\psi_h \cdot \nabla w_P \, dx = \int_\Omega f_h w_P dx + \int_{\Gamma_1} g_{1h} w_P d\Gamma \ \forall\ w_P \in \mathcal{B}_{oh} . \end{cases}$$

Using the expansion

$$(3.9)\quad \psi_h = \sum_{Q \in \Sigma_{oh}} \psi_h(Q) w_P + \sum_{Q \in \Sigma_h \cap \Gamma_o} g_o(Q) w_Q$$

of ψ_h, we can express (3.8) (and equivalently (3.5)) as a linear system in the $\psi_h(Q)$, $Q \in \Sigma_{oh}$, whose matrix is symmetric and positive definite ; this system is

$$(3.10)\quad \begin{cases} \sum_{Q \in \Sigma_{oh}} \psi_h(Q) \int_\Omega \nabla w_Q \cdot \nabla w_P dx = \int_\Omega f_h w_P dx + \int_{\Gamma_1} g_{1h} w_P dx - \sum_{Q \in \Sigma_h \cap \Gamma_o} g_o(Q) \int_\Omega \nabla w_Q \cdot \nabla w_P dx , \\ \text{for all } P \in \Sigma_{oh} . \end{cases}$$

To solve (3.10) we can use either direct methods (Gauss, Cholesky, etc...) or iterative (S.O.R., Conjugate Gradient with or without scaling, etc...).

From a computational point of view it is fairly easy to compute the right hand side and the matrix coefficients of the linear system (3.10) for the following reasons :

- Since w_P is piecewise linear $\forall P$, its gradient is a piecewise constant vector ;
- Since f_h and g_{1h} are piecewise linear, $f_h w_P$ and $f_h g_{1h}$ are piecewise quadratic.
- The support $\overline{\Omega}_P$ of w_P, where $\Omega_P = \{x | x \in \Omega,\ w_P(x) \neq 0\}$, consists of the union of those triangles of $\mathcal{T}_h$ with P as one of their vertices.

From these properties the various integrals required by (3.10) have to be done each time on a very small number of triangles and the integrand is, on each triangle, a low degree polynomial whose integration can be easily carried out exactly.

4. Formulation of the Grid Optimization problem.

The Grid Optimization problem will be considered for a family of triangulations with

- the same topology,
- the same number of vertices.

Some nodes playing an important role (for example, separation points between Γ_0 and Γ_1 or discontinuity points of g_1) are fixed. For computational purposes we have to number the nodes i.e. the vertices of $\mathcal{T}_h$. Let N_h = Card (Σ_h), then $\Sigma_h = \{P_i\}_{i=1}^{N_h}$; we denote by α_i, β_i the two coordinates of P_i (i.e. $P_i = \{\alpha_i, \beta_i\}$) and define $\underset{\sim}{\alpha}, \underset{\sim}{\beta} \in \mathbb{R}^{N_h}$ by

$$\underset{\sim}{\alpha} = \{\alpha_i\}_{i=1}^{N_h} \quad , \beta = \{\beta_i\}_{i=1}^{N_h} \quad .$$

We introduce now a subset E_f of $\mathbb{R}^{2N_h}$ consisting of the nodes corresponding to a given number of nodes and, possibly, several other conditions(some nodes are fixed, for example).

From $\{\underset{\sim}{\alpha}, \underset{\sim}{\beta}\} \in E_f$ we can define $\mathcal{T}_h$ and therefore the approximate problem

$$(4.1) \quad \begin{cases} \psi_h \in V_{gh} , \\ \int_\Omega \nabla\psi_h \cdot \nabla\phi_h \, dx = \int_\Omega f_h \phi_h dx + \int_{\Gamma_1} g_{1h}\phi_h d\Gamma \; \forall \; \phi_h \in V_{oh} ; \end{cases}$$

it means that the solution of (4.1) is in fact a function of $\{\underset{\sim}{\alpha}, \underset{\sim}{\beta}\}$, once f, g_o, g_1 are given.

Following Mc NEICE-MARCAL [6] we consider the Grid Optimization problem below

$$(4.2) \qquad \underset{\{\underset{\sim}{\alpha}, \underset{\sim}{\beta}\} \in E_f}{\text{Min}} \; \frac{1}{2} \int_\Omega |\nabla(\psi_h - \psi)|^2 dx$$

where in (4.2), ψ is the solution of the continuous problem and where the discrete solution ψ_h is a function of $\underset{\sim}{\alpha}, \underset{\sim}{\beta}$ through (4.1).

The above problem is a nonlinear, non-convex programming problem ; we shall not discuss in this report the question of existence and uniqueness which is a non trivial one (in fact the existence property alone is not difficult to prove, provided that E_f is "small enough", since in that case compactness techniques can usually be used).

The numerical solution of (4.2) which is a non trivial problem is considered in Sec. 5.

5. - ITERATIVE SOLUTION OF THE OPTIMIZATION PROBLEM.

In this section we shall suppose for simplicity that $f_h = f$, $g_{1h} = g_1$ and that $g_o = 0$ (then $V_{gh} = V_{oh}$). All these assumptions can be easily satisfied for the example in Sec. 2.

5.1. Reformulation of the Grid Optimization problem.

As mentioned in Sec. 1, the fact that ψ is not known can be a difficulty ; actually it is not the case for the minimization problem (4.2). We have first

$$(5.1)\quad \frac{1}{2}\int_\Omega |\nabla(\psi_h-\psi)|^2 dx = \frac{1}{2}\int_\Omega |\nabla\psi_h|^2 dx - \int_\Omega \nabla\psi_h\cdot\nabla\psi\, dx + \frac{1}{2}\int_\Omega |\nabla\psi|^2 dx\ .$$

From the above assumptions on f, g_o, g_1 and from (2.2), (4.1) we have

$$(5.2)\quad \int_\Omega \nabla\psi\cdot\nabla\psi_h dx = \int_\Omega f\psi_h\, dx + \int_{\Gamma_1} g_1\psi_h d\Gamma = \int_\Omega |\nabla\psi_h|^2 dx\ .$$

From (5.1), (5.2) we obtain

$$(5.3)\quad \frac{1}{2}\int_\Omega |\nabla(\psi_h-\psi)|^2 dx = -\frac{1}{2}\int_\Omega |\nabla\psi_h|^2 dx + \frac{1}{2}\int_\Omega |\nabla\psi|^2 dx\ .$$

Since $\int_\Omega |\nabla\psi|^2 dx$ is independent of $\mathcal{C}_h$, the minimization problem (4.2) can equivalently be written

$$(5.4)_1\quad \underset{\{\underset{\sim}{\alpha},\underset{\sim}{\beta}\}\in E_f}{\mathrm{Min}} \left\{-\frac{1}{2}\int_\Omega |\nabla\psi_h|^2 dx\right\}$$

or (from (5.2))

$$(5.4)_2\quad \underset{\{\underset{\sim}{\alpha},\underset{\sim}{\beta}\}\in E_f}{\mathrm{Min}} \left\{-\int_\Omega f\psi_h dx - \int_{\Gamma_1} g_1\psi_h d\Gamma\right\}\ ,$$

where in $(5.4)_1$, $(5.4)_2$, ψ_h is a function of $\underset{\sim}{\alpha},\underset{\sim}{\beta}$ through (4.1). We observe that ψ does not occur in $(5.4)_1$, $(5.4)_2$.

5.2. <u>On the calculation of the derivative of the cost function.</u>

In view of using descent methods (like steepest descent or conjugate gradient) it is of fundamental importance to have at our disposal the derivative, i.e. the gradient, of the cost function with respect to $\{\underset{\sim}{\alpha},\underset{\sim}{\beta}\}$.

We just consider the cost function in $(5.4)_1$, since the second case can be treated in a similar way. Let define therefore

$$j : \mathcal{O} \to \mathbb{R} ,$$

(where $\mathcal{O}$ is an open set of $\mathbb{R}^{2N_h}$ containing E_f) by

$$(5.5) \quad \begin{cases} j(\underset{\sim}{\alpha},\underset{\sim}{\beta}) = -\frac{1}{2}\int_\Omega |\nabla\psi_h|^2 dx , \\ \psi_h \text{ function of } \underset{\sim}{\alpha},\underset{\sim}{\beta} \text{ through (4.1).} \end{cases}$$

We have

$$(5.6) \quad \begin{cases} \delta j = \frac{\partial j}{\partial \underset{\sim}{\alpha}}\cdot\delta\underset{\sim}{\alpha} + \frac{\partial j}{\partial \underset{\sim}{\beta}}\cdot\delta\underset{\sim}{\beta} = -\int_\Omega \nabla\psi_h\cdot\nabla\delta\psi_h \, dx - \frac{1}{2}\frac{\partial}{\partial\underset{\sim}{\alpha}}\left(\int_\Omega |\nabla\psi_h|^2 dx\right)\cdot\delta\underset{\sim}{\alpha} - \\ \qquad - \frac{1}{2}\frac{\partial}{\partial\underset{\sim}{\beta}}\left(\int_\Omega |\nabla\psi_h|^2 dx\right)\cdot\delta\underset{\sim}{\beta} \; ; \end{cases}$$

we also have by differentiation of (4.1)

$$(5.7) \quad \begin{cases} \int_\Omega \nabla\delta\psi_h\cdot\nabla\phi_h dx + \frac{\partial}{\partial\underset{\sim}{\alpha}}\left(\int_\Omega \nabla\psi_h\cdot\nabla\phi_h dx\right)\cdot\delta\underset{\sim}{\alpha} + \frac{\partial}{\partial\underset{\sim}{\beta}}\left(\int_\Omega \nabla\psi_h\cdot\nabla\phi_h dx\right)\cdot\delta\underset{\sim}{\beta} = \\ = \frac{\partial}{\partial\underset{\sim}{\alpha}}\left(\int_\Omega f\,\phi_h dx\right)\cdot\delta\underset{\sim}{\alpha} + \frac{\partial}{\partial\underset{\sim}{\beta}}\left(\int_\Omega f\,\phi_h dx\right)\cdot\delta\underset{\sim}{\beta} + \frac{\partial}{\partial\underset{\sim}{\alpha}}\left(\int_{\Gamma_1} g_1\,\phi_h d\Gamma\right) \bullet \approx + \\ + \frac{\partial}{\partial\underset{\sim}{\beta}}\left(\int_{\Gamma_1} g_1\phi_h d\Gamma\right)\cdot\delta\underset{\sim}{\beta}, \quad \forall\phi_h \in V_{oh} . \end{cases}$$

Taking then $\phi_h=\psi_h$ in (5.7), we obtain from (5.6), (5.7) that

$$\left\{\begin{aligned} \delta j = {} & \frac{1}{2}\frac{\partial}{\partial \underset{\sim}{\alpha}}\left(\int_\Omega |\nabla\psi_h|^2 dx\right)\cdot\delta\underset{\sim}{\alpha} + \frac{1}{2}\frac{\partial}{\partial \underset{\sim}{\beta}}\left(\int_\Omega |\nabla\psi_h|^2 dx\right)\cdot\delta\underset{\sim}{\beta} + \\ & - \frac{\partial}{\partial \underset{\sim}{\alpha}}\left(\int_\Omega f\psi_h dx\right)\cdot\delta\underset{\sim}{\alpha} - \frac{\partial}{\partial \underset{\sim}{\beta}}\left(\int_\Omega f\psi_h dx\right)\cdot\delta\underset{\sim}{\beta} - \frac{\partial}{\partial \underset{\sim}{\alpha}}\left(\int_{\Gamma_1} g_1\psi_h d\Gamma\right)\cdot\delta\underset{\sim}{\alpha} - \\ & - \frac{\partial}{\partial \underset{\sim}{\beta}}\left(\int_{\Gamma_1} g_1\psi_h d\Gamma\right)\cdot\delta\underset{\sim}{\beta} , \end{aligned}\right.$$

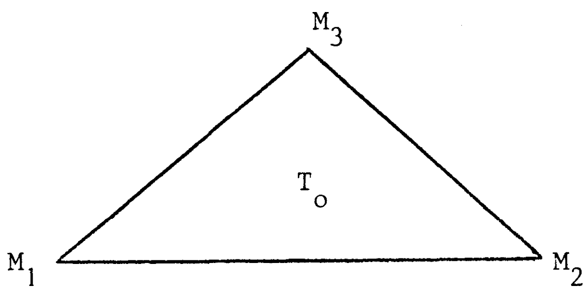
which implies that

$$(5.8)_1 \quad \frac{\partial j}{\partial \underset{\sim}{\alpha}} = \frac{1}{2}\frac{\partial}{\partial \underset{\sim}{\alpha}}\left(\int_\Omega |\nabla\psi_h|^2 dx\right) - \frac{\partial}{\partial \underset{\sim}{\alpha}}\left(\int_\Omega f\psi_h dx\right) - \frac{\partial}{\partial \underset{\sim}{\alpha}}\left(\int_{\Gamma_1} g_1\psi_h dx\right) ,$$

$$(5.8)_2 \quad \frac{\partial j}{\partial \underset{\sim}{\beta}} = \frac{1}{2}\frac{\partial}{\partial \underset{\sim}{\beta}}\left(\int_\Omega |\nabla\psi_h|^2 dx\right) - \frac{\partial}{\partial \underset{\sim}{\beta}}\left(\int_\Omega f\psi_h dx\right) - \frac{\partial}{\partial \underset{\sim}{\beta}}\left(\int_{\Gamma_1} g_1\psi_h dx\right) .$$

Obtaining $\frac{\partial j}{\partial \alpha}$ and $\frac{\partial j}{\partial \beta}$ from $(5.8)_1$, $(5.8)_2$, once ψ_h is known, is a painful task, but without theoretical difficulty. In view of the numerical treatment of the example of Sec. 2, we shall suppose that $f=0$, $g_1=$ const. and give more details about the calculation of the above derivatives.

Let M_1, M_2, M_3 be three points of $\mathbb{R}^2$, vertices of a <u>triangle</u> ; we suppose that the triangle $M_1M_2M_3$ is a <u>positive</u> triangle (see Fig. 5.1) denoted by T_o in the sequel.



<u>Figure 5.1</u>

Let $M_i = \{a_i, b_i\}$, $i=1,2,3$; if ϕ is a polynomial of degree ≤ 1 defined on T_0 we use the notation $\phi_i = \phi(M_i)$, $i=1,2,3$.
It is then quite easy to prove that

$$(5.9)_1 \qquad \frac{\partial\phi}{\partial x_1} = \frac{1}{2m(T_o)} \{\phi_1(b_2-b_3)+\phi_2(b_3-b_1)+\phi_3(b_1-b_2)\} ,$$

$$(5.9)_2 \qquad \frac{\partial\phi}{\partial x_2} = \frac{1}{2m(T_o)} \{\phi_1(a_3-a_2)+\phi_2(a_1-a_3)+\phi_3(a_2-a_1)\}$$

where $m(T_o)$, which is the measure of T_o, is given by

$$(5.10)\quad \begin{cases} 2m(T_o) = a_1(b_2-b_3)+a_2(b_3-b_1)+a_3(b_1-b_2) = \\ \qquad = b_1(a_3-a_2)+b_2(a_1-a_3)+b_3(a_2-a_1). \end{cases}$$

We obtain from $(5.9)_1$, $(5.9)_2$

$$(5.11)\qquad |\nabla\phi|^2 = \frac{1}{(2m(T_o))^2}(\phi_1\overrightarrow{M_2M_3}+\phi_2\overrightarrow{M_3M_1}+\phi_3\overrightarrow{M_1M_2})^2$$

where $\vec{V}^2 = \vec{V}\cdot\vec{V}$, i.e. the inner product of $\vec{V}$ with itself. We have then from (5.11)' since $\nabla\phi$ is constant over T_0,

$$(5.12)\qquad \int_{T_o}|\nabla\phi|^2dx = \frac{1}{2}\frac{1}{(2m(T_o))}(\phi_1\overrightarrow{M_2M_3}+\phi_2\overrightarrow{M_3M_1}+\phi_3\overrightarrow{M_1M_2})^2 .$$

Let $\phi_h \in V_h$ we have then from (5.12)

$$(5.13)\qquad \int_\Omega |\nabla\phi_h|^2dx = \frac{1}{2}\sum_{T\in\mathcal{C}_h}\frac{1}{2m(T)}(\phi_{1T}\overrightarrow{P_{2T}P_{3T}}+\phi_{2T}\overrightarrow{P_{3T}P_{1T}}+\phi_{3T}\overrightarrow{P_{1T}P_{2T}})^2$$

where $m(T)$ = measure of T and where P_{iT}, $i=1,2,3$ are the vertices of T in such a way that $P_{1T}P_{2T}P_{3T}$ is a positive triangle ; we set $\phi_h(P_{iT}) = \phi_{iT}$, $i=1,2,3$.

It follows then from $(5.9)_1$, $(5.9)_2$, (5.10), (5.13) that the function

$$\{\underset{\sim}{\alpha},\underset{\sim}{\beta}\} \to \int_\Omega |\nabla\phi_h|^2dx$$

is a rational function of the α_i,β_i whose partial derivatives are easy to compute from the above formulae. We observe also that, in the expansion (5.13)' the coordinates of a given node occur only for those triangles with that node as a vertex ; this property implies that most of the terms in the right and side of (5.13) do not contain the corresponding α_i,β_i and therefore their derivatives with respect to these parameters vanish.

Since we supposed f=0 we just have to consider now the calculation of $\frac{\partial}{\partial \alpha}\left(\int_{\Gamma} \psi_h dx\right)$, $\frac{\partial}{\partial \beta}\left(\int_{\Gamma} \psi_h dx\right)$; consider, as on Figure 5.2, a part of Γ_1 between the two-fixed nodes $A(=P_j)$ and $B(=P_{j+r})$.

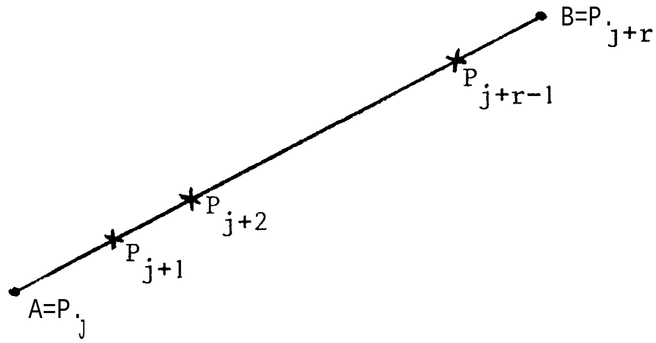


Figure 5.2

This edge AB is supported by a line whose equation is $x_2 = mx_1+\gamma$; we suppose, as on Fig. 5.2, that $0 < m < +\infty$. We have then, $\forall \phi_h \in V_h$, and with $\phi_k = \phi(P_k)$

$$(5.14) \quad \int_{\vec{AB}} \phi_h \, d\Gamma = \frac{(1+m^2)^{1/2}}{2} \sum_{k=j}^{j+r-1} (\alpha_{k+1}-\alpha_k)(\phi_{k+1}+\phi_k)$$

from which we obtain that for $k=j+1,\dots,j+r-1$

$$(5.15) \quad \frac{\partial}{\partial \alpha_k}\left(\int_{\vec{AB}} \phi_h \, d\Gamma\right) = \frac{(1+m^2)^{1/2}}{2} (\phi_k - \phi_{k+1}) \quad .$$

We can also use the β_k as independent variables (if AB is not supported by an horizontal line).

5.3. A conjugate gradient algorithm for solving the Optimization Problem.
Usually the Grid Optimization problem can be reduced to a Non Linear Programming problem of the following type

$$(5.16) \quad \begin{cases} \text{Find } u \in \mathbb{R}^N \text{ such that} \\ j(u) \leq j(v) \quad \forall v \in \mathbb{R}^N \end{cases}$$

where $j : \mathbb{R}^N \to \mathbb{R}$ is a C^1 functional. Let us describe a conjugate gradient algorithm with scaling ; we choose a Polak-Ribière type algorithm (see POLAK [7]) since it seems to be more efficient that the Fletcher-Reeves variant (cf. POLAK [7] again). In the sequel S is a symmetric, positive definite matrix, the scaling matrix ; the algorithm is defined as follows

(5.17) $u^o \in \mathbb{R}^N$ given,

(5.18) $g^o = \nabla j(u^o)$,

(5.19) $r^o = S^{-1} g^o$

(5.10) $w^o = r^o$

then for $n \geq 0$ with u^n, w^n known

(5.21) $\begin{cases} \text{find } \rho_n \in \mathbb{R} \text{ such that} \\ j(u^n - \rho_n w^n) \leq j(u^n - \rho w^n) \; \forall \rho \in \mathbb{R}, \end{cases}$

(5.22) $u^{n+1} = u^n - \rho_n w^n$

(5.23) $g^{n+1} = \nabla j(u^{n+1})$

(5.24) $r^{n+1} = S^{-1} g^{n+1}$

(5.25) $\gamma_{n+1} = \dfrac{(S\, r^{n+1}, r^{n+1} - r^n)}{(S\, r^n, r^n)} = \dfrac{(g^{n+1}, r^{n+1} - r^n)}{(g^n, r^n)}$,

(5.26) $w^{n+1} = r^{n+1} + \gamma_{n+1} w^n$.

In (5.25)' $(\cdot, \cdot)$ denotes the usual scalar product of $\mathbb{R}^N$.

The convergence of (5.17)-(5.26) is studied in [ ] (for S=I) where sufficient conditions for the convergence are given. In the particular case of problem (4.2) i.e. the Grid Optimization problem we have used S=I, but we have the feeling that taking for S an operator which is the discrete analogous of a suitable differential operator can be beneficial for the convergence ; the precise choice of such an operator is not

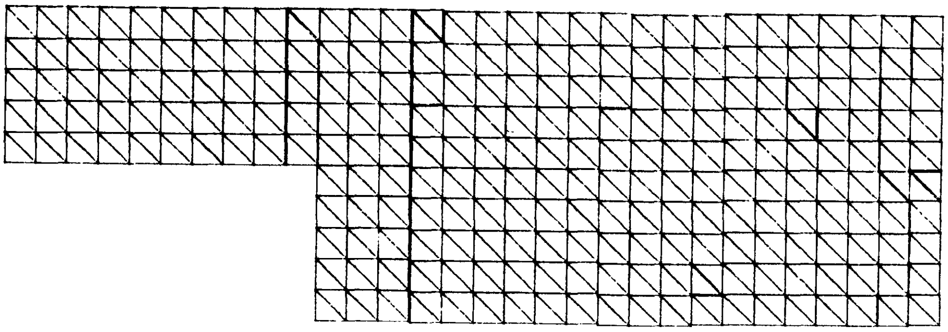


Figure 6.2

The Initial Grid

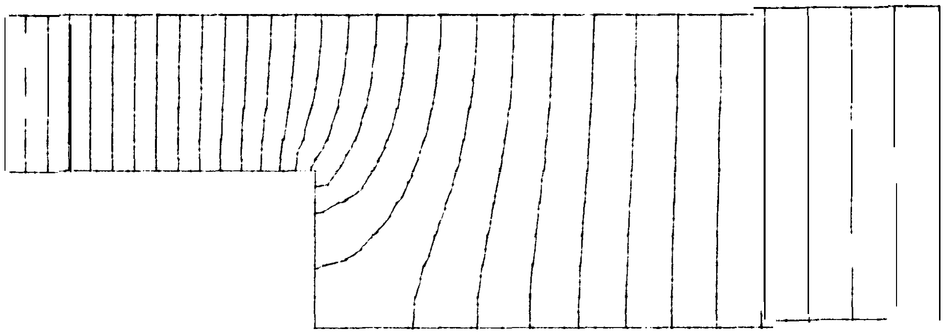


Figure 6.3

Equipotential lines

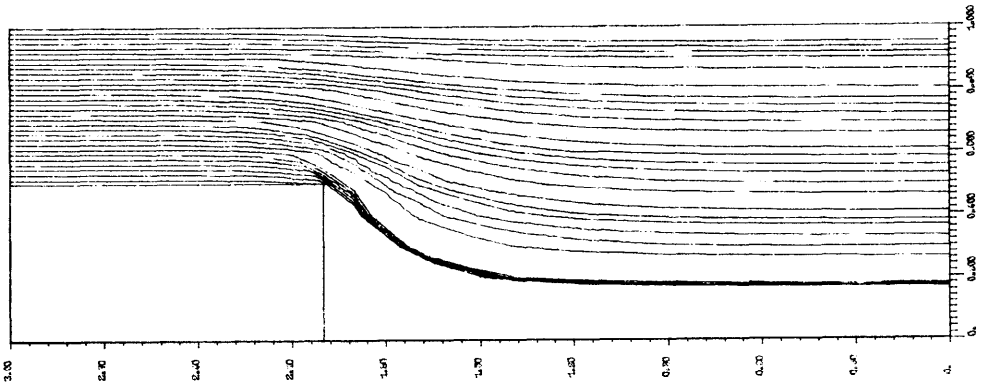


Figure 6.4

Stream lines

clear at the moment but several alternatives based on discrete second ordre elliptic operators are under consideration.

Concerning the one-dimensional problem (5.21) we can use a dichotomy or Fibonacci method.
In the case of the Grid Optimization problem we have to observe that each cost function evaluation requires the solution of a discrete elliptic problem (namely (4.1)) which is by itself a non trivial task.

6. - NUMERICAL EXPERIMENTS.

6.1. Description of the test problem.

We consider a test problem following the example of Sec. 2. We took for Ω the domain of Fig. 6.1,

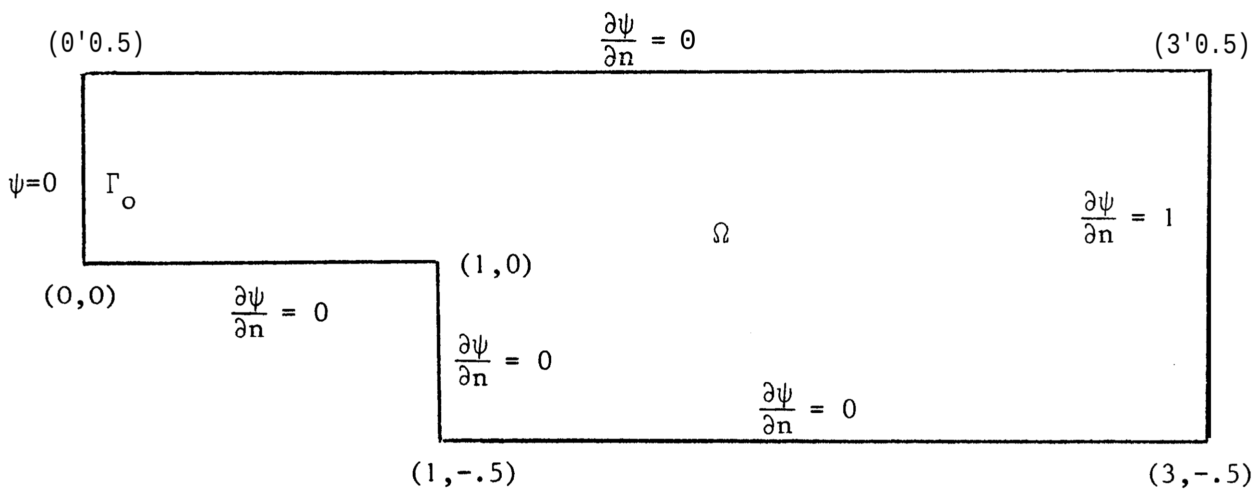


Figure 6.1

We have also shown on Figure 6.1 the boundary conditions ; in Ω we have

(6.1) $\Delta\psi = 0.$

The initial triangulation is shown on Fig. 6.2 and contains 500 triangles and 291 nodes. We suppose that the nodes on Γ_0, and also the corners are fixed, but we can move all the other nodes.

We have shown on Fig. 6.3, 6.4 the equipotential lines and the stream lines, respectively.

6.2. Numerical results.

We have used the conjugate gradient algorithm of Sec. 5.3 to obtain the optimal grid ; the problem under consideration is a Non-Linear Programming problem with 562 variables. The discrete elliptic problem (4.1) we have to solve several times at each iteration has 285 unknowns ; to solve it we have used a direct method based on Cholesky factorization.

As mentioned before we have used S=I ; the stationarity of the iterates is obtained in approximately 50 iterations and we have shown on Figures 6.5, 6.6, 6.7 the computed optimal grid and the corresponding equipotential and stream lines.

The computational time for 59 iterations is 17 minutes on the computer CII IRIS 80 ; this time include everything (i.e. printing, plotting, etc...). It is clear that the most demanding part is the Cholesky factorization required by each cost function evaluation. We think that using a conjugate gradient method scaled by a constant matrix can improve substantially the computational time.

From the optimal computed grid (shown on Fig. 6.5) we observe close to the re-entrant corner a stretching phenomenon along the stream lines for the triangles of $\mathcal{T}_h$; far enough of this corner the triangulation is not modified. Finally the optimal $\mathcal{T}_h$ behaves at what can be expected from intuition.

7. - CONCLUSION.

We-have considered in this report a procedure for computing the optimal grid in the finite element approximation of an elliptic test problem. The extension to more complicated problems, involving non linearities is still an open problem ; however following also the ideas developped in MARROCCO-PIRONNEAU [8] and BEGIS-GLOWINSKI [9] similar techniques have proved to be very useful for solving free boundary problems and also optimum design problems in which the unknown is for example the domain itself. At the moment, in collaboration with Avions Marcel-Dassault/ Bréguet Aviation, we are involved in an active research problem to compute airfoil or wing section of optimal shape for the potential flow of a compressible inviscid fluid in the transonic range, the flow being modelled by the full potential transonic equation.

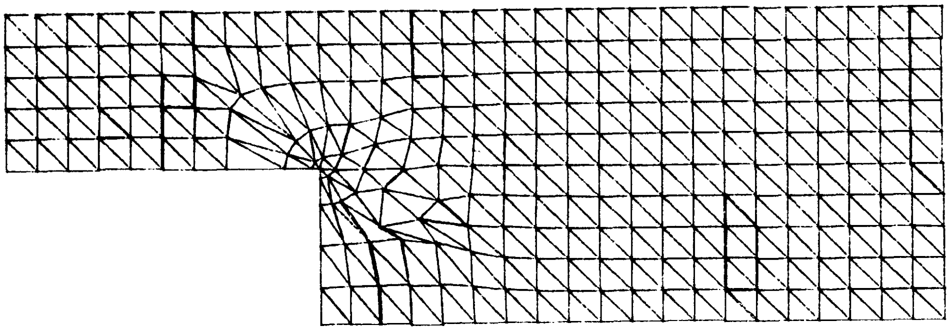


Figure 6.5

The computed Optimal Grid

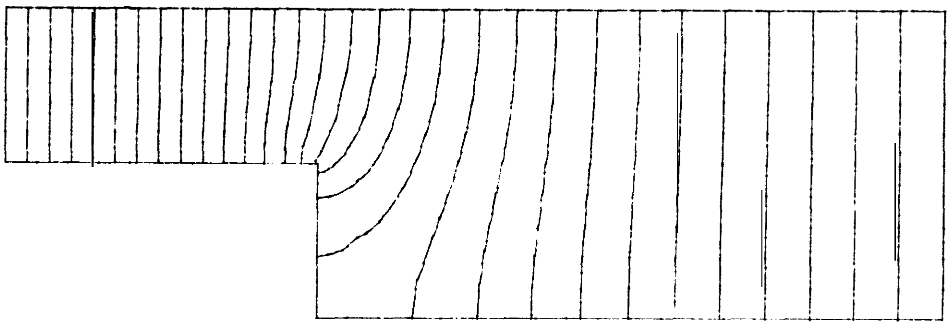


Figure 6.6

Equipotential lines

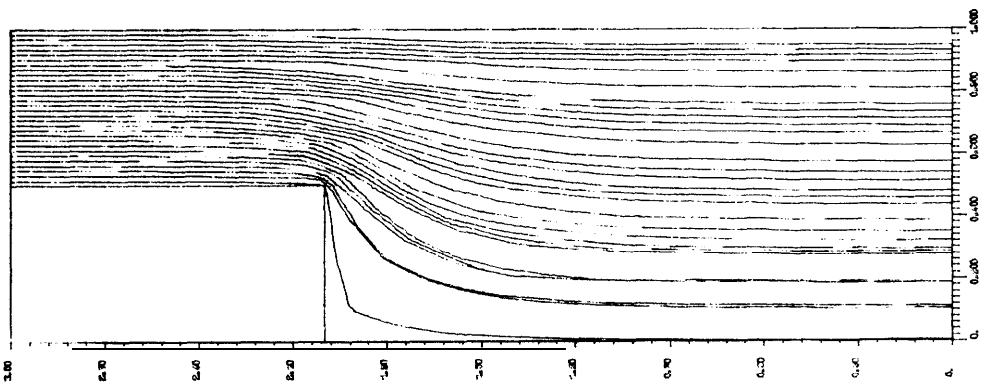


Figure 6.7

Stream lines.

BIBLIOGRAPHY

[1] LIONS J.L., MAGENES E. : Problèmes aux limites non homogènes et applications, Vol. 1, Dunod, Paris, 1968.

[2] NECAS J. : Les méthodes directes en théorie des equations elliptiques. Masson, Paris, 1967.

[3] ODEN J.T., REDDY J.N. : Mathematical Theory of finite elements. John Wiley and Sons, New-York, 1976.

[4] CIARLET P.G. : The finite element methods for elliptic problems. North-Holland, Amsterdam, 1978.

[5] STRANG G., FIX G. : An analysis of the finite element method. Prentice Hall, N.J., 1973.

[6] Mc NEICE G.M., MARCAL P.V. : Optimization of finite element grids based on miminum potential energy. Transactions of the Amer. Soc. Mech. Engrs., 95 Ser. B, N° 1, (1973), pp. 186-190.

[7] POLAK E. : Computational methods in Optimization. Acad. Press, New-York, 1971.

[8] MARROCCO A., PIRONNEAU O. : Optimum design with lagrangian finite elements : design of an electromagnet. Computer Methods Applied Mech. Eng., 15, (1978), pp. 277-308.

[9] BEGIS D., GLOWINSKI R. : Application de la méthode des elements finis à la resolution numérique d'un problème de domaine optimal. Méthodes de resolution des problèmes approchés. Applied Math. Optim., 2, (1975), pp. 130-169.