TWO RESULTS CONCERNING MULTICOLORING

by

V. Chvatal, M. R. Garey and D. S. Johnson

STAN-CS-76-582
DECEMBER 1976

COMPUTER SCIENCE DEPARTMENT
School of Humanities and Sciences
STANFORD UNIVERSITY

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER	STAN-CS-76-582
2. GOVT ACCESSION NO.	
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle)	TWO RESULTS CONCERNING MULTICOLORING.
5. TYPE OF REPORT & PERIOD COVERED	technical, December 1976
6. PERFORMING ORG. REPORT NUMBER	STAN-CS-76-582
7. AUTHOR(s)	V. Chvátal, M. R. Garey [redacted] D. S. Johnson
8. CONTRACT OR GRANT NUMBER(s)	N00014-76-C-0330; NSF-MCS-72-03752-A03
9. PERFORMING ORGANIZATION NAME AND ADDRESS	Stanford University Computer Science Department Stanford, Ca. 94305
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS	
11. CONTROLLING OFFICE NAME AND ADDRESS	Office of Naval Research Department of the Navy Arlington, VA 22217
12. REPORT DATE	Dec[redacted]76
13. NUMBER OF PAGES	12
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	ONR Representative: Philip Surra Durand Aeronautics Bldg., Rm. 165 Stanford University Stanford, Ca. 94305
15. SECURITY CLASS. (of this report)	Unclassified
15a. DECLASSIFICATION/DOWNGRADING SCHEDULE	

16. DISTRIBUTION STATEMENT (of this Report)

Releasable without limitations on dissemination

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

analysis of algorithms, combinatorial mathematics

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The m-chromatic number $X_m(G)$ of a graph $G = (V,E)$ is the least integer k such that there exists a mapping $f: V \rightarrow \{S \subseteq \{1,2,\ldots k\} : |S| = m\}$ having the property that $f(u) \cap f(v) = \phi$ whenever $\{u,v\} \in E$. This is a generalization of the standard notion of chromatic number and arises in connection with mobile telephone frequency assignments. Answering a question

of Lovász, our first result shows that for any $m \geq 1$ and any $\varepsilon > 0$, there exists a graph G for which $\chi_{m+1}(G)/\chi_m(G) > 2-\varepsilon$. This shows that the known bound of 2 for all m and G is essentially best possible. Our second result shows that the least integer m_0 for which $\chi_{m_0}(G)/m_0 = \lim_{m\to\infty} \chi_m(G)/m$ can be asymptotically as large as $e^{\sqrt{(n\log n)/2}}$ for some n vertex graphs, though it can never exceed $e^{(n\log n)/2}$.

TWO RESULTS CONCERNING MULTICOLORING

by

V. Chvátal
Stanford University
Stanford, California

and

M. R. Garey and D. S. Johnson
Bell Laboratories
Murray Hill, New Jersey 07974

ABSTRACT

The m-chromatic number $\chi_m(G)$ of a graph $G = (V,E)$ is the least integer k such that there exists a mapping $f:V \to \{S \subseteq \{1,2,\ldots,k\}: |S| = m\}$ having the property that $f(u) \cap f(v) = \phi$ whenever $\{u,v\} \in E$. This is a generalization of the standard notion of chromatic number and arises in connection with mobile telephone frequency assignments. Answering a question of Lovász, our first result shows that for any $m \geq 1$ and any $\varepsilon > 0$, there exists a graph G for which $\chi_{m+1}(G)/\chi_m(G) > 2-\varepsilon$. This shows that the known bound of 2 for all m and G is essentially best possible. Our second result shows that the least integer m_0 for which $\chi_{m_0}(G)/m_0 = \lim_{m\to\infty} \chi_m(G)/m$ can be asymptotically as large as $e^{\sqrt{(n\log n)/2}}$ for some n vertex graphs, though it can never exceed $e^{(n\log n)/2}$.

This research was supported in part by National Science Foundation grant MCS 72-03752 A03 and by the Office of Naval Reserch contract N00014-76-C-0330.

TWO RESULTS CONCERNING MULTICOLORING

by

V. Chvátal
Stanford University
Stanford, California

and

M. R. Garey and D. S. Johnson
Bell Laboratories
Murray Hill, New Jersey 07974

I. INTRODUCTION

The following generalization of the standard notion of graph coloring has been of recent interest [1,3,4,6,7,8]. A multicoloring of a graph $G = (V,E)$ is a function f defined on V whose values are sets (of "colors") satisfying $f(u) \cap f(v) = \phi$ whenever $\{u,v\} \in E$. For positive integers k,m, a (k,m)-coloring of $G = (V,E)$ is a multicoloring f of G such that $|f(v)| = m$ for each $v \in V$ and $|\bigcup_{v \in V} f(v)| = k$. The m-chromatic number $\chi_m(G)$ is the least integer k such that there exists a (k,m)-coloring of G. (This last definition differs from that of [6,7] by a factor of m.) Notice that for $m = 1$ these definitions correspond to the usual graph coloring notions. The purpose of this note is to resolve two questions about multicoloring conveyed to us by P. Erdös [2].

The first question deals with the relationship between $\chi_m(G)$ and $\chi_{m+1}(G)$. It is not difficult to see that

$$\chi_{m+1}(G) \le \chi_m(G) + \chi_1(G) \le 2 \cdot \chi_m(G)$$

with equality possible in the right-hand inequality only for $m = 1$. Lovász asked [2] whether, for each value of m, there exist graphs G such that $\chi_{m+1}(G) > (2-\epsilon)\chi_m(G)$. We shall answer this question in the affirmative.

The graphs we shall use are defined as follows: for positive integers $n \geq 2m$, the graph G_m^n has vertex set consisting of all m-element subsets of $\{1,2,\ldots,n\}$ and has an edge between two such vertices exactly when their intersection is empty. It is easy to see that $\chi_m\left(G_m^n\right) \leq n$ merely by considering the multicoloring provided by the definition of G_m^n and, in fact, it is proved in [7,8] that $\chi_m\left(G_m^n\right) = n$. Thus, to answer the question of Lovász, it suffices to prove the following theorem:

Theorem 1. For each $m \geq 2$, there exists a constant c such that for all sufficiently large n

$$\chi_{m+1}\left(G_m^n\right) \geq 2n - c.$$

In order to prove Theorem 1, we require the following lemma, which is an immediate consequence of a special case of Theorem 3 in [5].

Lemma 1. For fixed $m \geq 2$ and n sufficiently large, there exists a constant a_0 such that the number of m-element subsets of $\{1,2,\ldots,n\}$ which can be chosen so that no two are disjoint but there is no element common to all is at most $a_0 n^{m-2}$.

<u>Proof of Theorem 1</u>. Fix m. We merely need to show that, for all sufficiently large n,

$$\chi_{m+1}(G^n_m) - \chi_{m+1}\left(G^{n-1}_m\right) \geq 2$$

and the result will follow by induction. So suppose we have a $(k,m+1)$-coloring of G^n_m such that $k = \chi_{m+1}\left(G^n_m\right)$, where n is any integer sufficiently large that the conclusion of Lemma 1 holds and such that $\binom{n-1}{m-1} > ma_0n^{m-2}$, where a_0 is the constant of Lemma 1. We first claim that there must be at least n+1 colors which each appear on more than a_0n^{m-2} vertices.

Suppose there are n or fewer colors which each appear on more than a_0n^{m-2} vertices. By Lemma 1, each such color can appear on at most $\binom{n-1}{m-1} > a_0n^{m-2}$ vertices since they must all share a common element. Thus, since each of the $\binom{n}{m}$ vertices receives exactly m+1 colors, we must have

$$(m+1)\binom{n}{m} \leq n\binom{n-1}{m-1} + (k-n)a_0n^{m-2}$$

$$\leq m\binom{n}{m} + (k-n)a_0n^{m-2}$$

or, rewriting,

$$\binom{n}{m} \leq (k-n)a_0n^{m-2}$$

Since

$$k = \chi_{m+1}\left(G^n_m\right) \leq \chi_m\left(G^n_m\right) + \chi\left(G^n_m\right)$$

$$\leq 2\chi_m\left(G^n_m\right) = 2n$$

it follows that we must have

$$\binom{n}{m} \leq a_0 n^{m-1} .$$

However this is a contradiction, since n was chosen sufficiently large that $\binom{n}{m} = \frac{n}{m}\binom{n-1}{m-1} > \frac{n}{m} m a_0 n^{m-2} = a_0 n^{m-1}$, and the claim follows.

Thus there are at least n+1 colors which each appear on more than $a_0 n^{m-2}$ vertices. The set of vertices on which any color i appears must form a collection of pairwise-intersecting m-element subsets of $\{1,2,\ldots,n\}$, by definition of G_m^n. Thus, by Lemma 1, whenever color i appears on more than $a_0 n^{m-2}$ vertices, all those vertices must contain some common element e_i. Since there are more than n such colors, we must have $e_i = e_j$ for some i and j. If we delete from G_m^n all the vertices containing $e_i = e_j$, we obtain a copy of G_m^{n-1} and a (k-2,m+1)-coloring of it, since colors i and j have disappeared. Therefore

$$\chi_{m+1}\left(G_m^{n-1}\right) \leq k-2 = \chi_{m+1}\left(G_m^n\right) - 2$$

and the theorem is proved. ∎

The second question involves what we call the <u>ultimate multichromatic number</u> $\chi^*(G)$ defined by

$$\chi^*(G) = \inf_m \chi_m(G)/m.$$

It is proved in [1,7] that the value of $\chi^*(G)$ is always achieved for some finite m. One easy way to see this is to formulate the problem of determining $\chi^*(G)$ as a linear programming problem (as done in [4]): Let $v_1, v_2, \ldots, v_n$ be an ordering of the vertices of G and let $S_1, S_2, \ldots, S_\ell$ be an ordering of the independent sets of G. Define x_{ij} to be 1 whenever $v_i \in S_j$ and 0 otherwise. Then the value of $\chi^*(G)$ is given by

$$\chi^*(G) = \min \sum_{j=1}^{\ell} r_j$$

subject to: $r_j \geq 0, \quad 1 \leq j \leq \ell;$

$$\sum_{j=1}^{\ell} x_{ij} r_j = 1, \quad 1 \leq i \leq n.$$

One can show easily, using Hadamard's Theorem, that no basis matrix for this problem can have determinant exceeding $n^{n/2}$ and this is an upper bound on the value of m required.

This upper bound however seems ridiculously large. Erdös asked [2] (as did the authors, independently) whether $\chi^*(G)$ could always be achieved for an m not exceeding the number of vertices of G. We answer this in the negative, constructing graphs for which extremely large values of m are necessary to achieve $\chi^*(G)$.

Let C_p denote the graph which is a cycle on p vertices. The <u>join</u> G_1+G_2 of two graphs G_1 and G_2, having disjoint vertex sets, consists of all edges and vertices in the two given graphs along with all edges joining a vertex from G_1 to a vertex from G_2. We use the following two lemmas in our construction:

<u>Lemma 2</u>. [4,7] For all integers $p \geq 1$,

$$\chi^*(C_{2p+1}) = 2 + (1/p).$$

<u>Lemma 3</u>. [7] For all graphs G_1 and G_2,

$$\chi^*(G_1+G_2) = \chi^*(G_1) + \chi^*(G_2).$$

Let p_i denote the i^{th} prime and define the graph $G(i)$ to be $C_{2p_1+1} + C_{2p_2+1} + \cdots + C_{2p_i+1}$. The number of vertices n of G(i) is given by

$$n = i + 2 \sum_{j=1}^{i} p_j .$$

Applying Lemmas 2 and 3, we obtain

$$\chi^*(G(i)) = 2i + \sum_{j=1}^{i} (1/p_j) .$$

Since $\chi_m(G(i))$ must always be an integer, it follows that the least value of m for which $\chi^*(G(i)) = \chi_m(G(i))/m$ can be no less that $\prod_{j=1}^{i} p_j$ (and in fact that value of m will work). Using the Prime Number Theorem and expressing this lower bound in terms of n, we obtain the asymptotic lower bound of

$$e^{\sqrt{(n\log n)/2}} .$$

Thus, though this is still quite far from the upper bound of

$$n^{n/2} = e^{(n\log n)/2}$$

we see that extremely large values of m <u>can</u> be required in order to achieve $\chi^*(G)$.

REFERENCES

[1] F. H. Clarke and R. E. Jamison, "Multicolorings, Measures and Games on Graphs", Discrete Math., 14 (1976), 241-245.

[2] P. Erdös, personal communication.

[3] M. R. Garey and D. S. Johnson, "The Complexity of Near-Optimal Graph Coloring", J. ACM 23 (1976), 43-49.

[4] E. N. Gilbert, "Mobile Radio Frequency Assignments", unpublished technical memorandum, 1972.

[5] A. J. W. Hilton and E. C. Milner, "Some Intersection Theorems for Systems of Finite Sets", Quart. J. of Math. (Oxford second series), 18 (1967), No. 72, 369-384.

[6] A. J. W. Hilton, R. Rado, and S. H. Scott, "A (<5)-color theorem for Planar Graphs, Bull. London Math. Soc. 5 (1973), 302-306.

[7] S. H. Scott, Multiple Node Colorings of Finite Graphs, Ph.D. thesis, Dept. of Mathematics, University of Reading (England), 1975.

[8] S. Stahl, "n-Tuple Colorings and Associated Graphs", J. Combinatorial Theory-Series B, 20 (1976), 185-203.