# THEORETICAL AND PRACTICAL ASPECTS OF SOME INITIAL-BOUNDARY VALUE PROBLEMS IN FLUID DYNAMICS

by

Joseph Oliger and Arne Sundström

STAN-CS-76-578
NOVEMBER 1976

COMPUTER SCIENCE DEPARTMENT
School of Humanities and Sciences
STANFORD UNIVERSITY

# THEORETICAL AND PRACTICAL ASPECTS OF SOME INITIAL-BOUNDARY VALUE PROBLEMS IN FLUID DYNAMICS

by

Joseph Oliger and Arne Sundström

## ABSTRACT

Initial-boundary value problems for several systems of partial differential equations from fluid dynamics are discussed. Both rigid wall and open boundary problems are treated. Boundary conditions are formulated and shown to yield well-posed problems for the Eulerian equations for gas dynamics, the shallow-water equations, and linearized constant coefficient versions of the incompressible, anelastic equations. The "primitive" hydrostatic meteorological equations are shown to be ill-posed with any specification of local, pointwise boundary conditions. Analysis of simplified versions of this system illustrates the mechanism responsible for ill-posedness.

# THEORETICAL AND PRACTICAL ASPECTS OF SOME INITIAL-BOUNDARY VALUE PROBLEMS IN FLUID DYNAMICS

by

Joseph Oliger* and Arne Sundström**

## 0. Introduction

There is now considerable interest in initial-boundary value problems for various systems of partial differential equations arising in fluid dynamics. This interest stems, primarily, from efforts to create useful computational models of various processes for the purposes of prediction (atmospheric processes, ocean circulation, etc.) 'and the detailed study of various phenomena (convection, flow in wind tunnels, lee waves, eddies, etc.). Such calculations are not new. As these computational models have become more accurate difficulties with the boundary conditions have become more evident. This has led first to the examination of the various discretizations used and then back to the differential equations whose approximate solutions are sought.

Such a backward sequence of events may seem surprising. Naturally, the initial-boundary value problems for the differential equations should have been carefully examined first since we cannot expect our approximations to be reasonable if they approximate a problem which does not have reasonable solutions. The reason it has gone this way is clear.

*Computer Science Department, Stanford University, Stanford, California [illegible]. Supported in part by the Office of Naval Research under contract N00014-75-C-1132 and the National Science Foundation under grant DCR75-13497.

**National Defence Research Institute, 10450 Stockholm 80, Sweden.

It is natural to first examine the evidence where it appears and, as usual, the computations have been ahead of the analysis. The initial-boundary value problems for these systems of differential equations are not easy to analyze; and, in fact, adequate tools for a rather complete analysis have only recently become available stemming from the work of Kreiss [12,13]

The current interest has resulted in several works based on the classical energy method (e.g., Elvius and Sundström [9], Davies [5,6], de Rivas [ 7 ] and Dutton [8] ), which follow the earlier work of Serrin [17], Sundström [19] and Campbell [1]. However, this method only works for a limited class of equations and boundary conditions. Some authors have, unfortunately, made unallowable assumptions (over-specification of boundary conditions, omission of terms, etc.) in futile attempts to make their problems fit into this class. We will discuss some instances of this in detail. This seems to be a real hazard in the use of the energy method since the effects of such assumptions are often well-buried in intermediate estimates and consequently overlooked.

We begin with a general discussion of well-posedness followed by a review of properties of the adiabatic, inviscid Eulerian equations of fluid dynamics (system A). We then study two approximations of the Eulerian equations: the hydrostatic "primitive equations" of meteorology (system B1) and the incompressible, anelastic equations (system B2). Finally, we discuss the shallow-water, or barotropic, equations (system C) which can be considered as a further simplification of system B1 or B2. It is interesting to consider these equations in this order so that the effect of each successive approximation can be observed. The systems A and C are symmetrizable, hyperbolic systems but systems B1 and B2 are not hyperbolic. These facts have profound influence on the well-posedness of initial boundary-value problems for these systems.

We consider two types of boundary conditions which arise naturally in many situations. Most of our analysis will deal with certain quarter-space problems but we will always have the following underlying situation. Let $\Omega \subset R^2$ be an open, connected region with smooth boundary, $\partial\Omega$, and $\bar{\Omega} = \Omega \cup \partial\Omega$. We will consider the system C on the domain $D_1 = \bar{\Omega} \times [0,T]$ and the systems A, B1 and B2 on the domain $D_2 = \bar{\Omega} \times I \times [0,T]$ where $I = [0,\infty)$ or $I = [0,1]$. The two types of boundary conditions we consider on $\partial\Omega$ are: (1) rigid wall boundaries and (2) open boundaries. The rigid wall case corresponds to a physical situation which requires the normal velocity to vanish at the boundary and is the simpler of the two types. This situation is often encountered in oceanography. Open boundaries occur in limited area forecasting, wind tunnel flow, and studies of small scale or local phenomena in meteorology and oceanography. Open boundaries do not arise from a natural physical situation and a suitable form for the boundary conditions is not obvious. Boundary conditions which do not introduce boundary layer phenomena are usually wanted in this case. That is, these boundary conditions should determine the interior flow as though, in fact, the boundaries were not there at all. In each case we give necessary conditions for the form of the boundary conditions in order that the problems be well-posed. We will also give particular boundary conditions which yield well-posed problems. We show that systems A, B2, and C can be treated satisfactorily and that system B1 is ill-posed for local, pointwise boundary conditions. For the linearized, constant coefficient versions of systems B1 and B2, well-posed boundary conditions are given. It seems reasonable to conjecture that these boundary conditions also yield a well-posed problem for system B2. However, the corresponding boundary conditions for system B1 cannot be easily implemented for the general problem.

As already mentioned, many of the presently used boundary conditions specify more data than is allowed. These specifications preclude the existence of smooth solutions except in very special, unrealistic situations where the exact solution is known on the boundary without error. Errors must, however, be expected in the boundary data arising from errors in measurement, the use of constant boundary values, or from computations over larger regions if some telescoping grid technique is used. We will discuss the implications of such overspecifications.

Most of our analysis deals with inviscid systems of equations. Viscous terms are added to these equations in many forecast integrations. They are often motivated physically as representing "eddy diffusion" of momentum and potential temperature. The effect of these terms on the main part of the solution is usually small. The real motive for including them is often non-physical. Since the equations are nonlinear, initiallongwave phenomena can produce shorter wave phenomena which cannot be accurately represented by the approximation used. To prevent resulting aliasing errors and nonlinear instability the computational method should be provided with a dissipative filter term, and the most primitive form of filter is just such an "eddy diffusion" term.

In both cases, the viscous coefficients are so small that we should expect the boundary conditions to be close to those valid for the corresponding inviscid system. The viscous equations do, however, require additional boundary conditions, and, as an effect, viscous boundary layers may occur at the boundaries. Such boundary layers may sometimes be appropriate, as in the rigid wall situation. However, at open boundaries, they are inappropriate. We shall therefore discuss the formulation of boundary conditions when viscous terms are included and show how these conditions can be chosen so that no singular boundary layers result as the cofficients of the viscous terms tend to zero.

## 1. Well-Posedness

Our main goal is to establish the existence or non-existence of certain a priori estimates, or energy inequalities, valid for the solutions of the various initial-boundary value problems under consideration here. In this section we discuss the form of these estimates and some implications that follow from them.

For the purposes of this discussion let us write our problems in the general form

$$
\begin{aligned}
&L\underset{\sim}{u} = \underset{\sim}{F} && \text{in } \bar{\Omega} \times I \times [0,T] && \text{(the differential equation)}\\
(1.1)\quad &\underset{\sim}{u}\big|_{t=0} = \underset{\sim}{u}_0 && \text{in } \bar{\Omega} \times I && \text{(the initial conditions)}\\
&\underset{\sim}{u}^{I} = S\underset{\sim}{u}^{II} + \underset{\sim}{g} && \text{in } \partial\Omega \times I \times [0,T] && \text{(the boundary conditions)}
\end{aligned}
$$

where L is a partial differential operator; $\underset{\sim}{u}$, $\underset{\sim}{F}$ and $\underset{\sim}{u}_0$ are vector functions of dimension k, $\underset{\sim}{u} = (\underset{\sim}{u}^{I}, \underset{\sim}{u}^{II})^t$; $\underset{\sim}{u}^{I}$ and $\underset{\sim}{g}$ are of dimension $\ell$, $u^{II}$ is of dimensionsion $k-\ell$; and S is a real $\ell \times (k-\ell)$ matrix. For a linear, first order hyperbolic equation in $R^3$, L would take the form

$$
(1.2)\qquad L = \frac{\partial}{\partial t} + \sum_{j=1}^{3} A_j(\underset{\sim}{x},t)\frac{\partial}{\partial x_j} + B(\underset{\sim}{x},t).
$$

The boundary conditions express the components $\underset{\sim}{u}^{I}$ of $\underset{\sim}{u}$ in terms of the components $\underset{\sim}{u}^{II}$ of $\underset{\sim}{u}$ and the given function $\underset{\sim}{g}$. The matrix S can be thought of as a generalized reflection operator. Changes of variable may be necessary to bring certain desired boundary conditions into the form given here but this does not introduce any essential restriction. The partitioning $\underset{\sim}{u} = (\underset{\sim}{u}^{I}, \underset{\sim}{u}^{II})^{t}$ is based upon the characteristic variables, or Riemann invariants, of the problem, that is, the components of $\underset{\sim}{u}^{I}$ can be called "incoming" quantities and those of $u^{II}$ can be called "outgoing" quantities. This partitioning of $\underset{\sim}{u}$ will be discussed in detail for the problems we consider.

The estimates we seek are of the form

$$(1.3)\qquad (\|\underset{\sim}{u}\|_{\Omega\times I\times[0,T]} + \delta\|\underset{\sim}{u}\|_{\partial\Omega\times I\times[0,T]} + \|\underset{\sim}{u}(T)\|_{\Omega})$$

$$\leq e^{KT}(\|\underset{\sim}{F}\|_{\Omega\times I\times[0,T]} + \|\underset{\sim}{g}\|_{\partial\Omega\times I\times[0,T]} + \|\underset{\sim}{u}_0\|_{\Omega\times I})$$

where the norms are $L^2$ norms or weighted $L^2$ norms over the regions indicated by their subscripts, $K > 0$ is a constant independent of T, and $\delta = 0$ or 1. We will refer to (1.3) with $\delta = 0$ as the weak form of (1.3). The differences in the properties of solutions $\underset{\sim}{u}$ which satisfy only the weak form (1.3) from those which satisfy (1.3) with $\delta$ -- 1 are discussed by Kreiss [13]. We will not belabor the distinction here and be satisfied with the

Definition 1.1. We will say that the problem (1.1) is well-posed if the estimate (1.3) holds for all solutions $\underset{\sim}{u}$ of (1.1) with $\underset{\sim}{F}$, $\underset{\sim}{u}_0$ and $\underset{\sim}{g}$ in $L^2$.

Uniqueness and stability with respect to perturbations in the data follow from the estimate (1.3). We refer to the works of Kreiss [12, 13], Majda and Osher [14] and Strikwerda [18] for detailed discussions of the particular weighted $L^2$ norms and the general theory for systems of hyperbolic and incompletely parabolic equations.

The equations we are considering are all quasi-linear. However, we can obtain our a priori estimates a posteriori over intervals $[0,T]$ when a smooth solution exists, i.e., we can consider coefficients $A_j(\underset{\sim}{u}(\underset{\sim}{x},t),\underset{\sim}{x},t)$ as functions of $\underset{\sim}{x},t$ if $\underset{\sim}{u}(\underset{\sim}{x},t)$ is known. Furthermore, iterations based upon the linearized variational form of the problems can be used to establish existence for those t-intervals where the iterations converge. We will not pursue this here, but rather assume the existence of smooth solutions over the interval of consideration. We must prescribe boundary conditions that do not preclude the existence of such smooth solutions. This is the case if too many conditions are specified. Too few conditions preclude uniqueness, of course.

The results by Kreiss [12] and Strikwerda [18] also show that the systems (A) and (C) are stable to perturbations by lower order terms. This implies that we need not consider the effects of terms such as undifferentiated frictional terms and coriolis forces in our analysis.

It is essentially due to this fact that the analysis of variable coefficient problems can be reduced to that of corresponding constant coefficient problems via the construction of appropriate pseudo-differential operators (Kreiss [12], Taylor [20], Majda and Osher [14], Strikwerda [18]). This stability property also allows us to reduce problems on our general domain $\Omega$ with smooth boundaries to families of quarter-plane problems by making local changes of coordinates such that, e.g., $\partial\Omega$ is mapped into $x_1 = 0$ and $\Omega$ into $x_1 \geq 0$. Such mappings only introduce new terms which are of lower order. More detail about this process can be found in Majda and Osher [14] and Strikwerda [18].

Existing theoretical results cover problems with a smooth non-characteristic boundary for classes of equations which include A and C and their modifications resulting from the inclusion of the usual eddy 'viscosity terms (Kreiss [12], Majda and Osher [14], Strikwerda [18]). Extensions to problems in regions with corners and uniformly characteristic boundaries have been studied by Majda and Osher [14]. However, the important case where the velocities change sign on the boundary and do not vanish in a neighborhood of such a boundary point is not covered by existing theory. This often occurs in the applications we consider, e.g., the solid-wall type of boundary conditions, and when the flow direction reverses to change an inflow or outflow section of the boundary to an outflow or inflow section, respectively. There must be characteristic points on any smooth boundary of a simple connected region with open boundaries which has both inflow and outflow sections of the boundary.

We cannot treat the influence of such points on $\partial\Omega$ here but conjecture that no important modifications are usually necessary for problems like those we treat here.

We will use both the classical energy method and Kreiss' normal mode analysis to establish the well-posedness of these problems. The solid-wall boundary problems are all treated using the energy method which provides us with estimates of the form (1.3) directly. Some boundary conditions for the open boundary problems can be treated in this way, but, in general, we must use normal mode analysis for these problems.

## 2. The Eulerian Questions (System A).

The basic hydrodynamic and thermodyanmic laws governing the motion of an adiabatic and inviscid fluid are given by the Eulerian equations

$$
(2.1) \qquad
\begin{aligned}
\frac{d}{dt}\underset{\sim}{u} + \alpha\nabla p + \underset{\sim}{F} &= 0 \\
\frac{d}{dt}\alpha - \alpha\nabla\cdot\underset{\sim}{u} &= 0 \\
\frac{d}{dt}p + p\gamma\nabla\cdot\underset{\sim}{u} &= 0
\end{aligned}
$$

where u is the three-dimensional velocity vector, $\underset{\sim}{u} = (u_1,u_2,u_3)^t$, $\alpha$ is the specific volume, and p the pressure of the fluid; $\gamma = c_p/c_v$ is the lapse rate of the fluid, $\underset{\sim}{F}$ represents zero-order and forcing terms, e.g., coriolis and gravity forces, and

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + u \cdot \nabla = \frac{\partial}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial}{\partial x_j} \quad .$$

In vector notation, the equations are

$$\frac{\partial}{\partial t} \underset{\sim}{q} + \sum_{j=1}^{3} A_j(\underset{\sim}{q}) \frac{\partial}{\partial x_j} \underset{\sim}{q} + \underset{\sim}{F}(\underset{\sim}{q}) = 0 \tag{2.3}$$

where $\underset{\sim}{q} = (u_1, u_2, u_3, \alpha, p)^t$, and

$$A_1 = \begin{pmatrix} u_1 & 0 & 0 & 0 & \alpha \\ 0 & u_1 & 0 & 0 & 0 \\ 0 & 0 & u_1 & 0 & 0 \\ -\alpha & 0 & 0 & u_1 & 0 \\ p\gamma & 0 & 0 & 0 & u_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} u_2 & 0 & 0 & 0 & 0 \\ 0 & u_2 & 0 & 0 & \alpha \\ 0 & 0 & u_2 & 0 & 0 \\ 0 & -a & 0 & u_2 & 0 \\ 0 & p\gamma & 0 & 0 & u_2 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} u_3 & 0 & 0 & 0 & 0 \\ 0 & u_3 & 0 & 0 & 0 \\ 0 & 0 & u_3 & 0 & a \\ 0 & 0 & -\alpha & u_3 & 0 \\ 0 & 0 & p\gamma & 0 & u_3 \end{pmatrix}$$

The matrices $A_j$ all have real eigenvalues $u_j$, $u_j$, $u_j$, $u_j + c$, and $u_j - c$, with distinct eigenvectors, $c = (p\gamma\alpha)^{1/2}$ is the sound speed of the fluid. The matrices are not symmetric but it is easy to find a symmetric, positive definite matrix

$$R = \alpha^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & a^2 p\gamma/\alpha & a^2 \\ 0 & 0 & 0 & a^2 & (1+a^2)\frac{a}{p\gamma} \end{pmatrix} = T^{*-1}T^{-1}$$

where $a^2$ is an arbitrary real, non-zero parameter such that the transformed matrices $T^{-1}A_jT$ are all symmetric. The system A is thus a quasi-linear system of hyperbolic partial differential equations, see e.g., Courant and Hilbert [4]. Since no closed-form expression for the solution to this system is known, a rigorous evaluation of the effects of different approximations, inhomogeneous terms, and boundary conditions is, in general, impossible. For the problems we are considering the solutions are usually continuous and smooth. The matrices $A_j$ and R are then also smooth. As long as the deviations $\underset{\sim}{q}'(\underset{\sim}{x},t)$ from the exact solution $\underset{\sim}{q}(\underset{\sim}{x},t)$ are small, they should then approximately satisfy the linearized variational equations

(2.4) $$\frac{\partial}{\partial t}\underset{\sim}{q}' + \sum_{j=1}^{3} A_j(\underset{\sim}{q}) \frac{\partial}{\partial x}\underset{\sim}{q}' + C\underset{\sim}{q}' + \underset{\sim}{F}' = 0$$

where $\underset{\sim}{q}' = (u_1', u_2', u_3', \alpha', p')^t$,

$$
C = \begin{pmatrix}
\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} & \frac{\partial p}{\partial x_1} & 0 \\
\frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} & \frac{\partial p}{\partial x_2} & 0 \\
\frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} & \frac{\partial p}{\partial x_3} & 0 \\
\frac{\partial \alpha}{\partial x_1} & \frac{\partial \alpha}{\partial x_2} & \frac{\partial \alpha}{\partial x_3} & -\sum_{j=1}^{3} \frac{\partial u_j}{\partial x_j} & 0 \\
\frac{\partial p}{\partial x_1} & \frac{\partial p}{\partial x_2} & \frac{\partial p}{\partial x_3} & 0 & \gamma \sum_{j=1}^{3} \frac{\partial u_j}{\partial x_j}
\end{pmatrix}
$$

and the term $\underset{\sim}{F}'$ can represent other low-order terms. This system is a linear hyperbolic system in $\underset{\sim}{q}'$, and the well-posedness can be studied by either the classical energy method or by Kreiss' normal mode analysis. In the energy method, the basic idea is to show that a suitable norm for $\underset{\sim}{q}'$ satisfies a growth equation of the form

$$\frac{\partial}{\partial t} \|q'\| \leq K\|\underset{\sim}{q}'\| + \|\underset{\sim}{F}'\| \tag{2.5}$$

where $\|.\|$ is an inner-product norm $\|\underset{\sim}{q}'\| = (\int_\Omega \underset{\sim}{q}'^* \underset{\sim}{M}\underset{\sim}{q}' \, dx_1 dx_2 dx_3)^{1/2}$ equivalent to the Euclidean $L^2$-norm $(\int_\Omega |q'|^2 \, dx_1 dx_2 dx_3)^{1/2}$. We can show the well-posedness of the pure initial-value or Cauchy problem in the $L^2$-norm for the Eulerian equations by choosing $M$ as the matrix R given above. From

$$(2.6)\quad \frac{\partial}{\partial t}(q'^{*}Rq')$$

$$= -\sum_{j=1}^{3}\frac{\partial}{\partial x_j}(q'^{*}RA_j q') + q'^{*}\{\frac{\partial R}{\partial t} + \sum_{j=1}^{3} RA_j - RC - C^{*}R\}q' - q'^{*}RF' - F'^{*}Rq'$$

we get

$$(2.7)\quad \frac{\partial}{\partial t}\int q'^{*}Rq'\, dx_1\, dx_2\, dx_3$$

$$= \int[q'^{*}\{\frac{\partial R}{\partial t} + \sum_{j=1}^{3}\frac{\partial}{\partial x_j}(RA_j) - RC - C^{*}R\}q' - q'^{*}RF' - F'^{*}Rq']dx_1 dx_2 dx_3$$

and since R, $RA_j$, etc., are bounded, slowly varying matrices, we can easily establish a growth equation of type (2.5).

This inequality and energy norm is essentially equivalent to that used by Serrin [17] in his uniqueness proof for compressible fluids. He wanted the estimates to be valid for large deviations $q'$, which complicated the structure of the proof. However, at one step in his calculations, (eq. 25), he had to make an assumption which essentially meant that the deviations must be small. Furthermore, in the analysis of the limited-area case, an over-specified set of boundary conditions was used on the inflow portion of the boundary, thereby invalidating this part of the proof. The computational effects of such an overspecification will be discussed later.

For the initial-boundary value problem, we can use the growth equation (2.6) as before if and only if the boundary integral provides a non-negative contribution, i.e., if

(2.8) $$\oint_{\partial\Omega} \underset{\sim}{q}'^{*} R A_n \underset{\sim}{q}' \, ds \geq 0$$

where $A_n = \sum_{j=1}^{3} A_j \underset{\sim}{n}^t \cdot \underset{\sim}{e}_j$, $\underset{\sim}{n}$ is the unit vector in the outward normal direction, and $\underset{\sim}{e}_j$ is the unit vector in the $x_j$-direction. Note that if the boundary conditions are such that this inequality is satisfied, then well-posedness is proved, otherwise no conclusions can be made.

The integrand $\underset{\sim}{q}'^{*} R A_n \underset{\sim}{q}'$ is a quadratic form in the five variables $u_1', u_2', u_3'$, $\alpha'$, and $p'$. However, the number of boundary conditions is only equal to the number of inward characteristics, that is, the number of negative eigenvalues of $A_n$, see Kreiss [12]. These boundary conditions must be such that the related combinations of $\underset{\sim}{q}'$ (the Riemann invariants) are given in terms of known quantities and combinations corresponding to outward characteristics.

The initial-boundary value problems for the Eulerian equations arise from two different situations which must be studied separately:

1. A solid-wall boundary. Here, the physical boundary condition is that the normal velocity $v_n = \sum_{j=1}^{3} u_j \underset{\sim}{n}^t \cdot \underset{\sim}{e}_j$ should vanish at the boundary. This condition is consistent with the number of inward characteristics (one). Since it also gives $\underset{\sim}{q}'^{*} R A_n \underset{\sim}{q}' = 0$ identically, the well-posedness of the initial-boundary value problem follows directly.
2. An "open boundary," or a boundary located in the interior of a body of fluid. In this case the normal velocity is non-zero on the boundary, except at certain points. For supersonic fluids, all quantities $u_1$, $u_2$, $u_3$, $\alpha$, and $p$ should be specified at inflow points, giving $\underset{\sim}{q}'^{*} R A_n \underset{\sim}{q}' = 0$. At outflow points, no boundary

conditions should be prescribed and $q'^{*} R A_n q'$ is always $\geq 0$. For subsonic fluids, $A_n$ has four negative eigenvalues with distinct eigenvectors at inflow points so that four quantities have to be specified. At outflow points, the number of negative eigenvalues and quantities to be prescribed is only one. In both cases no obvious physical boundary conditions are known. The characteristic combinations of $q'$ may be found from the eigenvectors of $A_n$, but it is easier to proceed as follows.

The quadratic form

$$q'^{*} R A_n q'$$

$$= u_n \alpha^{-1}\{u_n'^2 + u_{\perp 1}'^2 + u_{\perp 2}'^2 + a^2 \alpha'^2 p\gamma/\alpha + (1+a^2) p'^2 \alpha/p\gamma + 2a^2 \alpha' p'\} + 2u_n' p'$$

($u_{\perp 1}$ and $u_{\perp 2}$ are the velocity components in two orthogonal tangential directions) can be rearranged as

$$= u_n \alpha^{-1}\{u_{\perp 1}'^2 + u_{\perp 2}'^2 + a^2[\alpha'(p\gamma/\alpha)^{1/2} + p'(\alpha/p\gamma)^{1/2}]^2\}$$
$$+ \frac{1}{2}(c + u_n)\,\alpha^{-1}(u_n' + p'(\alpha/p\gamma)^{1/2})^2 - \frac{1}{2}(c - u_n)\,a^{-1}(u_n' - p'(\alpha/p\gamma)^{1/2})^2$$

As before, $c = (p\gamma\alpha)^{1/2}$ is the basic sound speed.

At an inflow ($u_n < 0$) boundary the characteristic combinations corresponding to negative eigenvalues of $A_n$ are thus $u_{\perp 1}'$, $u_{\perp 2}'$, $u_n' - p'(\alpha/p\gamma)^{1/2}$, and $a'(p\gamma/q)^{1/2} + p'(\alpha/p\gamma)^{1/2} = c\theta'/\theta$, where $\theta$ is the "potentialtemperature" $\theta = (p_0^{1-1/\gamma}/R)\,\alpha p^{1/\gamma}$. The four boundary conditions should give relations of the form

$$u'_{\perp 1} = a_1(u'_n + p'(\alpha/p\gamma)^{1/2}) ,$$

$$u'_{\perp 2} = a_2(u'_n + p'(\alpha/p\gamma)^{1/2}) ,$$

$$u'_n - p'(\alpha/p\gamma)^{1/2} = a_3(u'_n + p'(\alpha/p\gamma)^{1/2}),$$

$$a'(p\gamma/\alpha)^{1/2} + p'(\alpha/p\gamma)^{1/2} = a_4(u'_n + p'(\alpha/p\gamma)^{1/2})$$

for deviations from the basic solution.

Before studying specific examples of boundary conditions giving relations of this type, it should be noted that not all such conditions give well-posed problems. With the classical energy method, we can actually only prove well-posedness when

$$u_n(a_1^2 + a_2^2 + a^2a_4^2) + \frac{1}{2}(c - u_n)a_3^2 \leq \frac{1}{2}(c + u_n)$$

since only then is $\underset{\sim}{q}'^* RA_n q' \geq 0$ at these inflow points. In this expression $a^2$ is the arbitrary, real and positive parameter of the matrix R. If we want to investigate the well-posedness for other values of $a_1, a_2, a_3, a_4$, we have to use the normal mode analysis of Kreiss, see Oliger and Sundström [15].

The classical energy method certainly works if the boundary conditions are such that $a_1 = a_2 = a_3 = a_4 = 0$. We can obtain these relations by prescribing (at inflow points) the two tangential velocity components, the potential temperature $\theta$, and the combination $u_n - \frac{2}{\gamma-1}(p\gamma\alpha)^{1/2}$.

The inequality is not satisfied if we have $a_1 = a_2 = a_4 = 0$ and $a_3 = \pm 1$, that is, if we try to give the tangential velocity components, the potential temperature, and either the normal velocity or the pressure at these inflow points. A third possibility for which the energy method does not work is $a_1 = a_2 = 0$, $a_3 = -1$, and $a_4 = 1$, that is, if we give all three velocity components and the specific volume a. As shown in Oliger and Sundström [15] using the normal mode analysis technique, this last combination actually gives a well-posed problem.

At the outflow ($u_n > 0$) boundary one quantity should be prescribed. It should give a relation of the form

$$u_n' - p'(\alpha/p\gamma)^{1/2}$$

$$= b_1 u_{l1}' + b_2 u_{l2}' + b_3\left(u_n' + p'(\alpha/p\gamma)^{1/2}\right) + b_4\left(\alpha'(p\gamma/\alpha)^{1/2} + p'(\alpha/p\gamma)^{1/2}\right)$$

for the linearized variational equations. It is simplest to prescribe the normal velocity so that $b_1 = b_2 = b_4 = 0$, $b_3 = -1$. The well-posedness of this boundary condition follows immediately from the positivity of $\tilde{q}'^{*} R A_n \tilde{q}'$. We may instead prescribe the pressure p, corresponding to $b_1 = b_2 = b_4 = 0$, $b_3 = 1$, or actually any combination of $u_n$ and p such that $(u_n' - p'(\alpha/p\gamma)^{1/2})^2 (c-u_n) \le (u_n' + p'(\alpha/p\gamma)^{1/2})^2 (c+u_n)$.

## 3. Basic Approximate Forms of the Eulerian Equations (Systems B1 and B2)

Although the Eulerian equations are the fundamental system of equations for most fluid flow problems, they are often used in modified approximate forms. In fact, they have almost never been used in the complete form given in the last section for geophysical calculations. The reason for this is simply a matter of economics and time, which are not unrelated. To compute an accurate approximate solution of the Eulerian equations for a relatively small problem requires quite a lot of computer time. These equations are an extremely "stiff" system of hyperbolic equations with a wide range of eigenfrequeneies and characteristic phase velocities. The ratios between the largest and smallest eigenvalues of the coefficient matrices $A_j$ in (2.3) are often as large as $10^2$ or $10^4$. The high-frequency eigensolutions (sound-waves) are often absent in the initial data and the solution,but their presence in the set of eigensolutions imposes a severe upper limit for the time-step in explicit numerical integration procedures. Implicit techniques that do not suffer from this difficulty lead to data structures which are difficult to manage and systems of nonlinear equations that are expensive to solve.

A second special aspect of many geophysical problems is the strong balance between the gravitational and vertical pressure gradient forces which is responsible for the basic stratification of the atmosphere and oceans. The vertical acceleration terms are usually much less than $10^{-4}$ times either of these terms. Even if we first subtract the time-independent part of the pressure field, we must still know the specific volume and pressure extremely accurately in order to compute the time derivative of the vertical velocity with even moderate accuracy. Similar, but less extreme, balances exist in the other equations. In the two remaining equations of motion there is a near balance between the horizontal pressure gradient and coriolis terms, and in the continuity equation there is a near balance of the components of divergence. These relations are often summarized in the statement that the atmosphere is, to a large extent, not only quasi-hydrostatic but also quasi-geostrophic and quasi-nondivergent.

To obtain a reasonable computational problem we must either: (1) find a more efficient numerical integration procedure, or (2) modify the equations in such a way that the high-frequency solutions are eliminated. The first alternative leads to integration methods of implicit type. The nonlinearity of the resulting implicit system and the difficulty of incorporating the near balance of the equations have not been successfully dealt with so far. A strict version of the second alternative is actually even more difficult to construct, if we try to eliminate only the solution of sound-wave type and obtain a system which is still hyperbolic. The main difficulty stems from the fact that the eigenvectors corresponding to the large eigenvalues are different for the different $A_j$'s. The nonlinearity of the system and the effects of variable coefficients are further complications.

Various approximate versions of the Eulerian equations have been derived by intuition, scale analysis, energy conservation considerations, and experience. The near balance of the equations simplifies this approach considerably. We shall study two such approximate sets of equations: the hydrostatic equations and the incompressible anelastic system of equations. Unfortunately, in both cases, the hyperbolic character of the system is lost.

The hydrostatic system is derived from the Eulerian equations by neglecting the vertical acceleration terms in the third equation of motion. One so obtains the "hydrostatic equation"

$$\alpha \frac{\partial p}{\partial z} + g = 0 \,. \tag{3.1}$$

Here, and from now on, we use the notation z for $x_3$ as a distinguished vertical coordinate and w for $u_3$ as the vertical velocity. This approximation is extremely accurate for the large-scale motion of the atmosphere. The wide-spread use of the hydrostatic approximation actually led meteorologists to calling the resulting system "the primitive equations of motion." This was motivated by a comparison with the still more approximate "quasi-geostrophic" system, but the term "primitive" is certainly misleading.

The use of the hydrostatic approximation has several important consequences. First, we no longer have a prognostic equation for the vertical velocity. Second, to maintain the hydrostatic equilibrium,, the time-changes of $\alpha$ and p must be coordinated in such a way that a $(\partial p/\partial z)$ is constant. This means that the pressure at any point in the model atmosphere can be determined from the pressure at any reference level and the mass of the separating layer, the integral $\int_{z_0}^{z} \alpha^{-1} dz$.

With this approximation, we obtain the hydrostatic system, system B1,

$$(3.2)\qquad \begin{aligned} &(\frac{\partial}{\partial t} + \underset{\sim}{u}\cdot\nabla)\underset{\sim}{u}_H + \alpha\,\nabla_H p + \underset{\sim}{F}_H = 0 \\ &\alpha\frac{\partial p}{\partial z} + g = 0 \\ &(\frac{\partial}{\partial t} + \underset{\sim}{u}\cdot\nabla)\alpha - \alpha\nabla\cdot\underset{\sim}{u} = 0 \\ &(\frac{\partial}{\partial t} + \underset{\sim}{u}\cdot\nabla)p + p\gamma\nabla\cdot\underset{\sim}{u} = 0 \end{aligned}$$

where $\underset{\sim}{u}_H=(u_1,u_2)^t$, $\underset{\sim}{u} = (\underset{\sim}{u}_H,w)^t$, $\underset{\sim}{F}_H = (F_1,F_2)^t$, and $\nabla_H = (\partial/\partial x_1, \partial/\partial x_2)^t$. This is a much more complicated system than it may seem at first glance. The equations (3.2) are not a hyperbolic system. To show this we form the variational equations

$$(3.3)\qquad \begin{aligned} &(\frac{\partial}{\partial t} + \underset{\sim}{u}\cdot\nabla)\underset{\sim}{u}'_H + \alpha\nabla_H p' + \underset{\sim}{u}'\cdot\nabla\underset{\sim}{u}_H + \alpha'\nabla_H p + \underset{\sim}{F}'_H = 0 \\ &\alpha\frac{\partial p'}{\partial z} = -\alpha'\frac{\partial p}{\partial z} = \frac{g}{\alpha}\alpha' \\ &(\frac{\partial}{\partial t} + \underset{\sim}{u}\cdot\nabla)\alpha' - \alpha\nabla\cdot\underset{\sim}{u}' + \underset{\sim}{u}'\cdot\nabla\alpha - \alpha'\nabla\cdot\underset{\sim}{u} = 0 \\ &(\frac{\partial}{\partial t} + \underset{\sim}{u}\cdot\nabla)p' + p\gamma\nabla\cdot\underset{\sim}{u}' + \underset{\sim}{u}'\cdot\nabla p + p'\gamma\nabla\cdot\underset{\sim}{u} = 0\,. \end{aligned}$$

The corresponding constant coefficient problem has periodic eigensolutions $u'_1 = \hat{u}'_1\exp\{i(\nu t + \omega_1 x_1 + \omega_2 x_2 + \omega_3 z)\}$, etc., for large $\omega_1$ and $\omega_2$ if and only if

(3.4)
$$\det D \sim 0$$
where

$$D = \begin{pmatrix} i\tilde{\nu} & 0 & \frac{\partial}{\partial z} u_1 & \alpha i\omega_1 + \frac{\alpha^2}{g} p_{x_1} i\omega_3 \\ 0 & i\tilde{\nu} & \frac{a}{\partial z} u_2 & \alpha i\omega_2 + \frac{\alpha^2}{g} p_{x_2} i\omega_3 \\ -\alpha i\omega_1 (g + c^2 i\omega_3) & -\alpha i\omega_2 (g + c^2 i\omega_3) & g\alpha_z + c^2\alpha\omega_3^2 & -\alpha^2(1+\gamma) i\omega_3 \nabla\cdot \underset{\sim}{u} \\ p\gamma\alpha_{x_1} + \alpha p_{x_1} & p\gamma\alpha_{x_2} + \alpha p_{x_2} & p\gamma\alpha_z + \alpha p_z & \frac{a}{g} i\tilde{\nu}(g + c^2 i\omega_3) \end{pmatrix}$$

where $\tilde{\nu} = \nu + \sum_{j=1}^{2} u_j \omega_j + w\omega_3$. This determinant is a cubic polynomial in the modified eigenfrequency $\tilde{\nu}$. It is easy to see that the roots $\tilde{\nu}_j$, $j = 1,2,3$, of (3.4) have the asymptotic behavior

$$\tilde{\nu}_1 = \mathcal{O}(1)$$

and

$$\tilde{\nu}_j = \mathcal{O}\left(\frac{\omega_1}{\omega_3}\right) + \mathcal{O}\left(\frac{\omega_2}{\omega_3}\right), \quad j = 2,3,$$

as $\omega_k \to \infty$. Thus, the system admits solutions with arbitrarily large signal speeds and is not hyperbolic. We cannot use the general methods and results for hyperbolic systems to find a well-posed set of boundary conditions.

Davies [5] tried to avoid this unfortunate effect of the hydrostatic approximation using a direct energy method approach similar to the one used in Section 2 for the Eulerian equations. The absence of the terms coming from $dw/dt$ changes (2.6) to

(3.5) $$\frac{\partial}{\partial t}\,(\underset{\sim}{q}'^{*}_{H}\,R\underset{\sim}{q}'_{H})$$

$$= -\sum_{j=1}^{2}\frac{\partial}{\partial x_j}\,(\underset{\sim}{q}'^{*}_{H}RA_j\underset{\sim}{q}'_{H}) - \frac{\partial}{\partial z}\,(\underset{\sim}{q}'^{*}R\tilde{A}_3\underset{\sim}{q}') + \underset{\sim}{q}'^{*}_{H}\left(\frac{\partial R}{\partial t} + \frac{\partial}{\partial x_1}(RA_1) + \frac{\partial}{\partial x_2}(RA_2)\right)\underset{\sim}{q}'_{H}$$

$$+ \underset{\sim}{q}'^{*}\left(\frac{\partial}{\partial z}(R\tilde{A}_3) - R\tilde{C} - \tilde{C}^{*}R\right)\underset{\sim}{q}' - \underset{\sim}{q}'^{*}R\underset{\sim}{F}' - F'^{*}R\underset{\sim}{q}'$$

where $\underset{\sim}{q}'_H = (u_1',u_2',0,\alpha',\nu')^t$ and the matrices $\tilde{A}_3$ and $\tilde{C}$ are obtained by deleting the terms arising from $w(\partial w/\partial z)$ in the original third equation from the matrices $A_3$ and C, respectively. Integrating (3.5) over the region $\Omega$ we obtain

(3.6) $$\frac{1}{2}\frac{\partial}{\partial t}\|\underset{\sim}{q}'_H\|^2$$

$$\leq K\|\underset{\sim}{q}'\|^2 + \|\underset{\sim}{q}'\|.\|\underset{\sim}{F}'\| - \int_{\Omega}\frac{\partial}{\partial x_1}\,(\underset{\sim}{q}'^{*}_{H}RA_1\underset{\sim}{q}'_{H}) + \frac{\partial}{\partial x_2}\,(\underset{\sim}{q}'^{*}_{H}RA_2\underset{\sim}{q}'_{H})$$

$$+ \frac{\partial}{\partial z}\,(\underset{\sim}{q}'^{*}R\tilde{A}_3\underset{\sim}{q}')\,dx_1dx_2dz\ .$$

Davies tried to find a set of boundary conditions such that the contribution from the boundary integral is non-positive,proceeding in the same way as we did with the Eulerian system A. He first conjectured that the number of boundary conditions could be chosen equal to the number which are required for the Eulerian equations. However, this conjecture is false. If this numberofboundary conditions is used the solution cannot be expected to satisfy the hydrostatic relation at the boundary. The problem is overspecified and the existence of continuous solutions is precluded. There is a further problem with this approach. The energy method is based upon norm equivalences and the norms of $\underset{\sim}{q}'$ and $\underset{\sim}{q}'_H$ are not equivalent. To obtain an adequate energy estimate, a bound for the term $\underset{\sim}{q}'^*(\frac{\partial}{\partial z}(RA_3) - R\tilde{C} - \tilde{C}^*R)\underset{\sim}{q}'$ of (3.5) in terms of $\|\underset{\sim}{q}'_H\|$ instead of $\|\underset{\sim}{q}'\|$ is necessary, but this is not possible. Consequently, we cannot draw any conclusions about the well-posedness of these equations from the reduced energy equation (3.6) for either the initial boundary value problem or the Cauchy problem.

Since the energy method does not work, we now turn to the normal mode analysis technique.

Normal mode analysis of the general equations here is rather complicated. If the motion is essentially horizontal we may instead consider linearization about an underlying basic state $\bar{\alpha}(z)$, $\bar{p}(z)$ which satisfies $\bar{\alpha}\bar{p}_z + g = 0$. . Due to this simplification, we cannot establish sufficient conditions for well-posedness in our succeeding analysis. However, we will at least be able to establish some necessary conditions and we can also expect the variational equations to reflect the main properties of the system. We use the notation $\alpha = \bar{\alpha}(z) + a'$, $p + \bar{p}(z) + p'$ and write-the horizontal velocities as $\underset{\sim}{u}_H = \underset{\sim}{v} + \underset{\sim}{u}'_H$ where $\underset{\sim}{v} = (v_1, v_2)^t$ is constant. If we neglect all of the nonlinear terms in primed quantities, we obtain the approximate system

$$\frac{\partial}{\partial t} \underset{\sim}{u}'_H + \sum_{j=1}^{2} v_j \frac{\partial}{\partial x_j} \underset{\sim}{u}'_H + \bar{\alpha}\nabla_H p' + \underset{\sim}{F}' = 0$$

$$\bar{\alpha} \frac{\partial p'}{\partial z} - g \frac{\alpha'}{\bar{\alpha}} = 0$$

(3.7)

$$\frac{\partial}{\partial t} \alpha' + \sum_{j=1}^{2} v_j \frac{\partial}{\partial x_j} \alpha' - \bar{\alpha}\nabla\cdot\underset{\sim}{u}' + \alpha_z w' = 0$$

$$\frac{\partial}{\partial t} p' + \sum_{j=1}^{2} v_j \frac{\partial}{\partial x_j} p' + \bar{p}\gamma\nabla\cdot\underset{\sim}{u}' - \frac{gw'}{\bar{\alpha}} = 0 .$$

We can then transform (3.7) to obtain the following equations in $u_1'$, $u_2'$, and $\bar{\alpha}p'$,

$$\frac{\partial}{\partial t} u'_H + \sum_{j=1}^{2} v_j \frac{\partial}{\partial x_j} \underset{\sim}{u}'_H + \nabla_H(\bar{\alpha}p') + \underset{\sim}{F}' = 0$$

$$\left(\frac{\partial}{\partial t} + \sum_{j=1}^{2} v_j \frac{\partial}{\partial x_j}\right) L(\bar{\alpha}p') + \nabla_H\cdot\underset{\sim}{u}'_H = 0$$

where

$$L(\bar{\alpha}p') = - \frac{\bar{\alpha}}{g} \frac{\partial}{\partial z} \left( \frac{\bar{p}\gamma}{\bar{p}\gamma\bar{\alpha}_z - g} \frac{\partial}{\partial z} (\bar{\alpha}p') \right) = - \frac{\bar{\alpha}}{g^2} \frac{\partial}{\partial z} \left(\frac{\bar{\alpha}}{\sigma} \frac{\partial}{\partial z} (\bar{\alpha}p')\right)$$

and $\sigma = -g^{-1}\bar{\alpha}^{2} \frac{\partial}{\partial z} \ln \bar{\theta}$ is the "static stability" of the basic state. We will always assume $\sigma > 0$, i.e., we are only studying perturbations of a stable stratification. The boundary condition at z = 0 is

$w' = 0$ which implies $\bar{p}\gamma\alpha' + \bar{\alpha}p' = 0$. As an upper boundary condition we have $\bar{p} \to 0$ as $z \to \infty$. Our condition at $z = 0$ can also be written as $\frac{\partial}{\partial z}(\bar{\alpha}p') = g\sigma p'/\bar{\alpha}$ so, for arbitrary $q'$,

$$\int_0^\infty \bar{\alpha}^{-1} q' \, L(\bar{\alpha}p') \, dz = -\int_0^\infty q' \frac{\partial}{\partial z}\left(\frac{\bar{\alpha}}{g^2\sigma} \frac{\partial}{\partial z}(\bar{\alpha}p')\right) dz$$

$$= \int_0^\infty \frac{\bar{\alpha}}{g^2\sigma} \frac{\partial q'}{\partial zz} \frac{\partial}{\partial z}(\bar{\alpha}p')\,dz + \left(\frac{1}{g\bar{\alpha}} q'\bar{\alpha}p'\right)_{z=0}$$

which shows that the operator $L$ is self-adjoint and half-bounded. Therefore, the system (3.8) is separable. If we expand the variables $u'_H$ and $\bar{\alpha}p'$ in the eigenfunctions $F_\nu(z)$ of $L$, we then obtain, for each $\nu$,

$$\left(\frac{\partial}{\partial t} + \sum_{j=1}^{2} v_j \frac{\partial}{\partial x_j}\right) u'_{H(\nu)} + \nabla_H (\bar{\alpha}p')_{(\nu)} + F_{(\nu)} = 0$$

(3.9)

$$\left(\frac{\partial}{\partial t} + \sum_{j=1}^{2} v_j \frac{\partial}{\partial x_j}\right) \lambda_\nu (\bar{\alpha}p')_{(\nu)} + \nabla_H \cdot u'_{H(\nu)} = \ldots$$

where the $\lambda_\nu$ are the eigenvalues of $L$, i.e.,

(3.10) $$LF_\nu(z) = \lambda_\nu F_\nu(z) .$$

Since $\sigma > 0$, this eigenvalue problem is of Sturm-Liouville type and the eigenvalues $\lambda_\nu$ are distinct, positive, real numbers. For each value of $\nu$ the system (3.9) is hyperbolic and has the same form as the shallow-water equations to be discussed later. The characteristic velocities are $v_j$, $v_j + c_\nu$ and $v_j - c_\nu$ where $c_\nu = \lambda_\nu^{-1/2}$. Under standard atmospheric conditions $c_0 \approx 322$ m/s., $c_1 \approx 34$ m/s., $c_2 \approx 17$ m/s. and $c_\nu \to 0$ as $\nu \to \infty$, see Wiin-Nielsen [22]. From this it follows that, for small $\nu$, two of the characteristic velocities will be positive if $v_j > 0$ and one will be positive if $v_j \leq 0$. When $\nu$ is so large that $c_\nu < |v_j|$, then all three characteristic velocities are positive if $v_j > 0$ and negative if $v_j < 0$.

It follows that the appropriate number of boundary conditions for our simplified problems must be different for these two classes of eigensolutions. For those components with $c_\nu < |v_n|$, all variables should be prescribed if $v_n < 0$ (inflow) and no variables should be prescribed if $v_n > 0$ (outflow). For the other components, the appropriate number of boundary conditions is two if $v_n < 0$ and one if $v_n > 0$. Possible forms for these boundary conditions are examined later for the equivalent shallow-water equations. The simplest choice of inflow conditions is probably to give both velocity components, $\underset{\sim}{u}'_{H(\nu)}$, for all $\nu$ and $\bar{\alpha}p'_{(\nu)}$ for those values of $\nu$ with $c_\nu < |v_n|$. At outflow, $v_n > 0$, the normal velocity component can be given for those $\nu$ with $c_\nu > |v_j|$. In the special case of a solid-wall boundary there is only one class of eigensolutions, since $c_\nu > |v_n|$ for all $\nu$. The condition $u_n(z) = 0$ yields $u'_{n(\nu)} = 0$ for all $\nu$

which, as shown later, yields a well--posed problem for each $\nu$ and, consequently, for the entire system, We reiterate that these conclusions are only valid for our simplified version of the variational equations and that we have no proof of their validity for the complete system. However, the conclusion that the boundary conditions must be separated in terms of the vertical eigenfunctions is valid for the complete system since that system admits particular solutions of the type that we have discussed. Our inability to provide sufficient conditions for the complete system is essentially due to the fact that we cannot show that all solutions of the complete equations can be expressed in terms of the eigenfunctions of (3.12). It follows from this discussion that local, pointwise boundary conditions cannot yield a well-posed problem for the open boundary problem for the hydrostatic equations; well-posed problems can only be obtained (1) if the boundary conditions are formulated in terms of local eigenfunction expansions or (2) nonlocal boundary operators are used. We know of no successful formulation of the second type.

A convenient byproduct of the hydrostatic approximation is the possibility of using variables other than z as the vertical coordinate (e.g., pressure, p, potential temperature, $\theta$, etc. ). If pressure is used as the vertical coordinate the equations (3.2) become

$$\frac{d}{dt}\underset{\sim}{u}_H + \nabla_p\varphi + \underset{\sim}{F} = 0$$

$$\frac{d}{dt}\alpha + \frac{\omega\alpha}{p\gamma} = 0 \tag{3.11}$$

$$\nabla_p\cdot\underset{\sim}{u}_H + \frac{\partial\omega}{\partial p} = 0$$

$$\frac{\partial\varphi}{\partial p} + \alpha = 0$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \underset{\sim}{u}_H \cdot \nabla_p + \omega \frac{\partial}{\partial p} ,$$

$$\omega = \frac{dp}{dt} , \qquad \varphi = gz ,$$

and $\nabla_p$ denotes the horizontal gradient/divergence operator on constant pressure surfaces.

The p-system has the advantage that the region of integration has a limited vertical extent, $0 \leq p \leq p_s$, where $p_s$ is the surface pressure, instead of $0 \leq z < \infty$ for the original system. The meteorological data are also collected and analyzed as functions of pressure which simplifies the construction of initial data fields on constant pressure surfaces. One disadvantage is that the lower boundary condition, $w = 0$ at $z = 0$, becomes $d\varphi/dt = 0$ at the unknown surface $p = p_s(\underset{\sim}{x},t)$ where $\varphi = 0$. The usual way to overcome this difficulty is to prescribe

$$(3.12) \qquad \frac{d\varphi}{dt} = \left( \frac{\partial}{\partial t} + \underset{\sim}{u}_H \cdot \nabla_p + \omega \frac{\partial}{\partial p} \right) \varphi = 0$$

at a constant pressure surface $p = p_0$, usually chosen as 1000mb, instead of at $p = p_s$. The upper boundary condition simply becomes $\omega = 0$ at $p = 0$.

It is easy to show that the transformation to the p-system does not change the nonhyperbolic character of the equations, and again an upper bound cannot be found for the rate of growth of disturbances on the solution in any conventional energy norm. As before, we have to limit the detailed study to an approximate system of variational equations. These are given by

$$
(3.13) \quad
\begin{aligned}
&\left(\frac{\partial}{\partial t} + \sum_{j=1}^{2} v_j \frac{\partial}{\partial x_j}\right)\underset{\sim}{u}'_H + \nabla_p \varphi' + \underset{\sim}{F}' = 0 \\
&\left(\frac{\partial}{\partial t} + \sum_{j=1}^{2} v_j \frac{\partial}{\partial x_j}\right)\alpha' + \omega' \left(\frac{\partial\bar{\alpha}}{\partial p} + \frac{\bar{\alpha}}{p\gamma}\right) = 0 \\
&\nabla_p \cdot \underset{\sim}{u}'_H + \frac{\partial \omega'}{\partial p} = 0 \\
&\frac{\partial}{\partial p}\varphi' + \alpha' = 0
\end{aligned}
$$

. where $\bar{\alpha} = \bar{\alpha}(p)$. We are only considering the mean translatory part of the advection terms. The system (3.13) will retain the essential features of the complete variational equations if the solution is close to steady-state and the motion is quasi-horizontal.

If we eliminate $\omega'$ and $\alpha'$, the system (3.13) can be written in terms of $\underset{\sim}{u}'_H$ and $\varphi'$ as follows:

$$
(3.14) \quad
\begin{aligned}
&\left(\frac{\partial}{\partial t} + \sum_{j=1}^{2} v_j \frac{\partial}{\partial x_j}\right)\underset{\sim}{u}'_H + \nabla_p \varphi' + \underset{\sim}{F}' = 0 \\
&\left(\frac{\partial}{\partial t} + \sum_{j=1}^{2} v_j \frac{\partial}{\partial x_j}\right)L\varphi' + \nabla_p \cdot \underset{\sim}{u}'_H = 0
\end{aligned}
$$

where

$$L\varphi' = -\frac{\partial}{\partial p}\left(\sigma^{-1}\frac{\partial\varphi'}{\partial p}\right)$$

with

$$\sigma = -\left(\frac{\partial\bar{\alpha}}{\partial p} + \frac{\bar{\alpha}}{p\gamma}\right) .$$

To show that (3.14) is also a separable system we must show that the upper and lower boundary conditions are consistent. At $p = 0$ we have $\omega' = 0$ so $\sigma^{-1}(\partial\varphi'/\partial p) = 0$. At $p = p_0$ the condition (3.12) may be transformed, using the equation (3.13) for a', to obtain

$$(3.15) \qquad \left(\frac{\partial}{\partial t} + \sum_{j=1}^{2} v_j \frac{\partial}{\partial x_j}\right)\left(\varphi' - \frac{\bar{\alpha}\alpha'}{\sigma}\right) = 0$$

so that

$$\sigma^{-1}\frac{\partial\varphi'}{\partial p} + \bar{\alpha}^{-1}\varphi' = 0 \quad \text{at} \quad p = p_0$$

if this condition is satisfied initially. Then

$$\int_0^{p_0} q' L\varphi' dp = \int_0^{p_0} \sigma^{-1}\frac{\partial q'}{\partial p}\frac{\partial\varphi'}{\partial p}dp + (\bar{\alpha}^{-1}q'\varphi')_{p=p_0}$$

which shows that L is self-adjoint and half-bounded. We can now expand $\underset{\sim}{u}'_H$ and $\varphi'$ in the eigenfunctions of L. The simplified variational equations then become a family of hyperbolic systems,

$$\left(\frac{\partial}{\partial t} + \sum_{j=1}^{2} v_j \frac{\partial}{\partial x_j}\right) u'_{H(\nu)} + \nabla_p \varphi'_{(\nu)} + F'_{(\nu)} = 0$$

(3.16)

$$\left(\frac{\partial}{\partial t} + \sum_{j=1}^{2} v_j \frac{\partial}{\partial x_j}\right) \kappa_\nu \varphi'_{(\nu)} + \nabla_p \underset{\sim}{u}'_{(\nu)} = 0$$

where the $\kappa_\nu$ are the eigenvalues of L, i.e.,

(3.17) $$LG_\nu(p) = \kappa_\nu G_\nu(p)$$

with corresponding eigenfunctions $G_\nu(p)$.

Again, the horizontal boundary conditions must be formulated differently as the normal component of velocity is larger or smaller than $\bar{\kappa}_\nu^{1/2}$. All of the conclusions made for the z-coordinate system apply in this case.

We now consider the second approximation of the Eulerian equations of motion, the incompressible, anelastic system (B2). Approximations of this type have been used in many areas of theoretical and applied fluid mechanics. In their most simple form, the equations for an incompressible fluid are

(3.18)
$$\frac{d}{dt}\underset{\sim}{u} + \alpha_0 \nabla p = 0$$

$$\nabla \cdot \underset{\sim}{u} = 0$$

where $\alpha_0$ is a constant. This system is usually derived from basic physical considerations,but it can also be obtained from the complete

Eulerian equations by letting $\alpha$ approach a limiting constant value, $\alpha_0$. The more general anelastic approximation, often used in studies of convective systems in the atmosphere, is based upon the following assumptions: 1) the potential temperature of the fluid is nearly constant (we denote this constant value by $\bar{\theta}$ in our following discussion); 2) the pressure deviates only slightly from a hydrostatic stratification; and 3) the typical horizontal and vertical length scales are similar. If the characteristic length scale is much smaller than the "scale height," $c_p\bar{\theta}/g$, we obtain the system

$$\frac{d}{dt}\underset{\sim}{u}_H + \bar{\theta}\nabla_H\pi = 0$$

$$\frac{d}{dt} w + \bar{\theta}\frac{\partial\pi}{\partial z} - g\frac{\tilde{\theta}}{\bar{\theta}} = 0$$

$$\frac{d}{dt}\tilde{\theta} = 0 \tag{3.19}$$

$$\nabla\cdot\underset{\sim}{u} = 0$$

where $\tilde{\theta}$ is the deviation of the potential temperature from a basic state $\bar{\theta}$, $\pi = c_p((p/p_0)^{1-1/\gamma} - (\bar{p}/p_0)^{1-1/\gamma})$, $\bar{p}$ denotes the isentropic pressure profile corresponding to $\bar{\theta}$, and

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \underset{\sim}{u}_H\cdot\nabla_H + w\frac{\partial}{\partial z}\ .$$

We note that, expect for the equation for $\tilde{\theta}$, this system is essentially the same as (3.18). A third system of the same basic form is the Boussinesq system which is often used in oceanography.

The linearized variational equations corresponding to the system (3.19) are, deleting the small zero-order terms,

$$\frac{d}{dt}\underset{\sim}{u}'_H + \bar{\theta}\nabla_H \pi' = 0$$

(3.20)
$$\frac{d}{dt}w' + \bar{\theta}\frac{\partial \pi'}{\partial z} - g\frac{\theta'}{\bar{\theta}} = 0$$

$$\frac{d}{dt}\theta' + \underset{\sim}{u}'\cdot\nabla\tilde{\theta} = 0$$

$$\nabla\cdot\underset{\sim}{u}' = 0 .$$

The existence of nontrivial periodic eigensolutions of the form $\exp[i(\nu t + \omega_1 x_1 + \omega_2 x_2 + \omega_3 z)]$ is equivalent to the determinant condition

(3.21)
$$\det\begin{pmatrix} i\tilde{\nu} & 0 & 0 & \bar{\theta}i\omega_1 & 0 \\ 0 & i\tilde{\nu} & 0 & \bar{\theta}i\omega_2 & 0 \\ 0 & 0 & i\tilde{\nu} & \bar{\theta}i\omega_3 & -g/\bar{\theta} \\ \frac{\partial\tilde{\theta}}{\partial x_1} & \frac{\partial\tilde{\theta}}{\partial x_2} & \frac{\partial\tilde{\theta}}{\partial z} & 0 & i\tilde{\nu} \\ i\omega_1 & i\omega_2 & i\omega_3 & 0 & 0 \end{pmatrix} = 0$$

where

$$\tilde{\nu} = \nu + \sum_{j=1}^{2} u_j\omega_j + w\omega_3 .$$

Equation (3.21) is third order in $\nu$ and of fifth order in the $\omega_i$. If the lower order terms are neglected, then the resulting equation has $\nu = -\sum_{j=1}^{2} u_j \omega_j - w\omega_3$ as a triple root. Thus, the anelastic equations are not a hyperbolic system, but their eigensolutions have, to highest order, time dependent behavior which is like that of hyperbolic systems.

We cannot obtain sufficient conditions for well-posedness of the initial boundary value problem for time-singular systems like (3.20) using the normal node analysis technique. The theoretical justification is lacking at present. However, rigorous results on the necessary form of the boundary conditions can be obtained since it is clear that pathological solutions can be constructed via the normal node technique following Agmon's construction [12].

Analysis of the eigensolutions of (3.20) shows that four boundary conditions must be given at inflow parts of the boundary and that one condition must be given at outflow parts of the boundary. Furthermore, pathological behavior like that exhibited by solutions of the approximate system B1 is not present. It can be shown using the energy method that the physical boundary condition $u_n = 0$ for a rigid wall boundary yields a well-posed problem with an energy norm of type $(u_1'^2 + u_1'^2 + w'^2 + a^2\theta'^2)^{1/2}$.

For the constant coefficient problems (3.20), $L^2$ estimates for the well-posedness can be obtained directly using Fourier-Laplace transform techniques for the initial boundary value problem on a quarter-space;

$t \geq 0$; $x_1 \geq 0$; $-\infty < x_2, z < \infty$. It follows that these problems are well-posed if $\underset{\sim}{u}$ and $\theta$ are given at inflow and if $u_n$ is given at outflow. However, as mentioned above, the reduction of the general problem to quarter-space problems and the variable coefficient problems to constant coefficient problems via freezing arguments is not covered by existing theory

## 4. The Barotropic or "Shallow-Water" Equations (System C).

A third approximation to the Eulerian equations, the shallow-water equations which are our system C, may be written

$$\frac{d}{dt}\underset{\sim}{u}_H + \nabla_H \varphi + \underset{\sim}{F} = 0$$

(4.1)

$$\frac{d}{dt}\varphi + \varphi \nabla_H \cdot \underset{\sim}{u}_H = 0$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \underset{\sim}{u}_H \nabla_H ,$$

and $\underset{\sim}{u}_H$ and cp are functions of time, t, and the horizontal space coordinates, $x_1$ and $x_2$, $\underset{\sim}{F}$ represents any zero order or forcing terms such as, e.g., the coriolis forces. The variable $\varphi$, the geopotential, is always positive. In most geophysical problem, the flow is subsonic so that $\varphi > u_1^2 + u_2^2$, but the opposite relation holds for both supersonic flow and for some of the subsystems derived from the hydrostatic equations (B1) by separation of variables.

At least three different names exist for system C. In meteorological applications it is usually called "the primitive barotropic equations." In oceanography the most common name is "the shallow-water equations." This system accurately describes wave motion on the surface of a homogeneous fluid when the horizontal wave length is much longer than both the vertical scale of motion and the depth of the fluid. The vector form of system C is

$$\frac{\partial}{\partial t} \underset{\sim}{q} + \sum_{j=1}^{2} A_j(\underset{\sim}{q}) \frac{\partial}{\partial x_j} \underset{\sim}{q} + \underset{\sim}{F} = 0 \tag{4.2}$$

where $\underset{\sim}{q} = (u_1, u_2, \varphi)^t$,

$$A_1 = \begin{pmatrix} u_1 & 0 & 1 \\ 0 & u_1 & 0 \\ \varphi & 0 & u_1 \end{pmatrix}, \quad \text{and} \quad A_2 = \begin{pmatrix} u_2 & 0 & 0 \\ 0 & u_2 & 1 \\ 0 & \varphi & u_2 \end{pmatrix}.$$

The eigenvalues of $A_j$ are $u_j$, $u_j + c$ and $u_j - c$ with $c = \varphi^{1/2}$. They are all real and have distinct eigenvectors. The symmetric and positive definite transformation matrix

$$R = \begin{pmatrix} \varphi & 0 & 0 \\ 0 & \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} = T^{*-1}$$

simultaneously symmetrizes $A_1$ and $A_2$. Thus, the system C has retained (or regained) the basic property of the original Eulerian equations (system A) of being a quasi-linear system of hyperbolic equations.

As before, we are only discussing problems with smooth solutions. The basic properties of the system may then be found from the corresponding linearized variational equations

$$(4.3) \qquad \frac{\partial}{\partial t} \underset{\sim}{q}' + \sum_{j=1}^{2} A_j(\underset{\sim}{q}) \frac{\partial}{\partial x_j} \underset{\sim}{q}' + C\underset{\sim}{q}' + \underset{\sim}{F}' =$$

where $\underset{\sim}{q}$ is a solution of (4.2), and the small disturbance $\underset{\sim}{q}'(x_1,x_2,t)$ may be generated by the inhomogeneous term $\underset{\sim}{F}'$ or be caused by an initial disturbance $\underset{\sim}{q}'(x_1,x_2,0)$. The matrix C has the form

$$C = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & -f + \frac{\partial u_1}{\partial x_2} & 0 \\ f + \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & 0 \\ \frac{\partial \varphi}{\partial x_1} & \frac{\partial \varphi}{\partial x_2} & \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \end{pmatrix}$$

The matrix C, the coefficient matrices $A_1(\underset{\sim}{q})$ and $A_2(\underset{\sim}{q})$, and the transformation matrix $R(\underset{\sim}{q})$ are slowly varying in space and time. It is easy to show the well-posedness of the Cauchy problem in the $L^2$-norm by the classical energy method. This 'was done by Elvius and Sundström [9] and Davies [5]. They also showed the well-posedness of the initial-boundary value problem for some possible sets of boundary conditions using the energy method. However, as pointed out by de Rivas [7], Davies overspecified the boundary conditions in his paper. For this reason our discussion follows that of Elvius and Sundström.

The growth equation for the energy norm of $\underset{\sim}{q}'$ is

$$(4.4)\qquad \frac{\partial}{\partial t}\int_\Omega \underset{\sim}{q}'^* R\underset{\sim}{q}'\, dx_1\, dx_2$$

$$= \int_\Omega \underset{\sim}{q}'^* \left\{\frac{\partial R}{\partial t} + \sum_{j=1}^{2} \frac{\partial}{\partial x_j}(RA_j) - RC - C^* R\right\}\underset{\sim}{q}'\, dx_1\, dx_2$$

$$- \int_\Omega \underset{\sim}{q}'^* R\underset{\sim}{F}' + \underset{\sim}{F}'^* R\underset{\sim}{q}'\, dx_1 dx_2 - \int_{\partial\Omega} \underset{\sim}{q}'^* RA_n \underset{\sim}{q}'\, ds$$

Since

$$\frac{\partial R}{\partial t} + \sum_{j=1}^{2} \frac{\partial}{\partial x_j}(RA_j) - RC - C^* R$$

$$= -\begin{pmatrix} 2\varphi \dfrac{\partial u_1}{\partial x_1} & \varphi\left(\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1}\right) & 0 \\ \varphi\left(\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1}\right) & 2\varphi \dfrac{\partial u_2}{\partial x_2} & 0 \\ 0 & 0 & \dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_2}{\partial x_2} \end{pmatrix}$$

we can rewrite (4.4) as

$$(4.5)\qquad \frac{\partial}{\partial t}\int_\Omega \underset{\sim}{q}'^* R\underset{\sim}{q}'\, dx_1\, dx_2$$

$$= \int_\Omega - \left\{\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right)(\varphi(u_1'^2 + u_2'^2) + \varphi'^2) + \varphi\left(\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2}\right)(u_1'^2 - u_2'^2)\right.$$

$$\left. + 2\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) u_1' u_2')\right\} dx_1\, dx_2$$

$$- \int_\Omega \underset{\sim}{q}'^* R\underset{\sim}{F}' + \underset{\sim}{F}'^* R\underset{\sim}{q}'\, dx_1\, dx_2 - \int_{\partial\Omega} \underset{\sim}{q}'^* RA_n \underset{\sim}{q}'\, ds\ .$$

The first integral is bounded by

$$\int_\Omega \left(\left[\left(\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2}\right)^2 + \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)^2\right]^{1/2} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2}\right) (\varphi(u_1'^2 + u_2'^2) + \varphi'^2) dx_1 dx_2$$

if we can neglect the contribution from the inhomogeneous term and if the boundary conditions ensure that the integral $\oint \underset{\sim}{q}'^* R A_n \underset{\sim}{q}' ds \geq 0$ . The growth rate of $\int_\Omega \underset{\sim}{q}'^* R \underset{\sim}{q}' \, dx_1 \, dx_2$ is then bounded by the maximum value of the quantity

$$\left(\left(\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2}\right)^2 + \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)^2\right)^{1/2} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} \quad ,$$

the difference between the deformation and divergence of the basic flow. Results of this type may also be found for finite disturbances (which was not possible for the Eulerian equations) Without any linearization, the complete variational equations are

$$\frac{\partial}{\partial t} \underset{\sim}{q}' + \sum_{j=1}^{2} A_j(\underset{\sim}{q} + \underset{\sim}{q}') \frac{\partial}{\partial x_j} \underset{\sim}{q}' + C\underset{\sim}{q}' + F' = 0 \; .$$

Using the complete transformation matrix $R(\underset{\sim}{q} + \underset{\sim}{q}')$, we get

$$\frac{\partial}{\partial t} \int_\Omega \underset{\sim}{q}'^* R \underset{\sim}{q}' \, dx_1 dx_2$$

$$= \frac{a}{\partial t} \int_\Omega (\varphi + \varphi')(u_1'^2 + u_2'^2) + \varphi'^2 \, dx_1 dx_2$$

$$= - \int_\Omega \Bigg\{ (u_1 + u_2)[(\varphi + \varphi')(u_1'^2 + u_2'^2) + \varphi'^2]$$

$$+ (\varphi + \varphi') \left[ \left( \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} \right) (u_1'^2 - u_2'^2) \right.$$

$$\left. + 2\left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) u_1' u_2' \right] \Bigg\} dx_1 dx_2$$

$$+ \int_\Omega 29' \left( u_1' \frac{\partial \varphi'}{\partial x_1} + u_2' \frac{\partial \varphi'}{\partial x_2} \right) + \varphi'^2 \left( \frac{\partial u_1'}{\partial x_1} + \frac{\partial u_2'}{\partial x_2} \right) dx_1 dx_2$$

$$- \int_\Omega \underset{\sim}{q}'^* R \underset{\sim}{F}' + \underset{\sim}{F}'^* R \underset{\sim}{q}' \, dx_1 dx_2 - \int_{\partial\Omega} \underset{\sim}{q}'^* R A_n \underset{\sim}{q}' \, ds \ .$$

Since

$$\int_\Omega 2\varphi' \left( u_1' \frac{\partial \varphi'}{\partial x_1} + u_2' \frac{\partial \varphi'}{\partial x_2} \right) + \varphi'^2 \left( \frac{\partial u_1'}{\partial x_1} + \frac{\partial u_2'}{\partial x_2} \right) dx_1 dx_2$$

$$= \int_\Omega \frac{\partial}{\partial x_1} (u_1' \varphi'^2) + \frac{\partial}{\partial x_2} (u_2' \varphi'^2) \, dx_1 dx_2$$

$$= \oint_{\partial\Omega} u_n' \varphi'^2 \, ds \ ,$$

we again obtain a growth estimate

$$(4.6)\qquad \frac{\partial}{\partial t}\int_\Omega q'^* R q'\, dx\, dy$$

$$\le \max\left\{\left(\left(\frac{\partial u_1}{\partial x_1}-\frac{\partial u_2}{\partial x_2}\right)^2+\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)^2\right)^{1/2}-\frac{\partial u_1}{\partial x_1}-\frac{\partial u_2}{\partial x_2}\right\}$$

$$\times\int_\Omega q'^* R q'\, dx_1 dx_2$$

if $F' = 0$ and the boundary integral $\int_{\partial\Omega} q'^* RA_n q' - u_n'\varphi'^2\, ds \ge 0$.

Note that we now have a bound for the growth rate which is valid for disturbances of arbitrary size and only involves the deformation and divergence of the undisturbed flow. The solutions $q$ and $q + q'$ may thus be any pair of solutions to the shallow-water equations. For finite $q'$, the integral $\int_\Omega q'^* R q' dx_1 dx_2 = \int_\Omega (\varphi+\varphi')(u_1'^2+u_2'^2)+\varphi'^2 dx_1 dx_2$ is no longer the square of an $L^2$-equivalent norm for $q'$, but as long as $\varphi + \varphi'$ (the thickness of the fluid layer) is strictly positive we can apply the corresponding Liapunov theorems. Usually, $|\varphi'|$ is so much smaller than $|\varphi|$ that the difference between $\left(\int_\Omega q'^* R(q+q')q'\, dx_1 dx_2\right)^{1/2}$ and the energy norm $\left(\int_\Omega q'^* R(q)q'\, dx_1 dx_2\right)^{1/2}$ is negligible.

All these estimates are valid if and only if the contribution from the boundary is strictly nonpositive. For the linearized variational equations we had to require $\int_{\partial\Omega} q'^* RA_n q' ds \ge 0$, i.e.,

$$\int_{\partial\Omega} v_n(\varphi(u_n'^2 + u_\perp'^2) + \varphi'^2) + 2\varphi u_n'\varphi' ds$$

$$= \int_{\partial\Omega} \{u_n\varphi u_\perp'^2 + \frac{\varphi}{2}(c+u_n)(u_n' + \varphi'/c)^2 - \frac{\varphi}{2}(c-u_n)(u_n' - \varphi'/c)^2\}\, ds \ge 0$$

For the complete variational equations, the integral is instead

$$\oint_{\partial\Omega} (u_n + u_n') [(\varphi + \varphi')(u_n'^2 + u_\perp'^2) + \varphi'^2] + (2\varphi + \varphi') u_n' \varphi' \, ds$$

$$= \int_{\partial\Omega} \{ (u_n + u_n')(\varphi + \varphi') u_\perp'^2$$

$$+ \frac{1}{2}(\varphi + \varphi')((\varphi + \frac{1}{2}\varphi')/c + u_n + u_n')(u_n' + \varphi'/c)^2)$$

$$- \frac{1}{2}(\varphi + \varphi')((\varphi + \frac{1}{2}\varphi')/c - u_n - u_n')(u_n' - \varphi'/c)^2 \} \, ds$$

where now $c = (\varphi + \varphi')^{1/2}$. We again study the two different types of boundary conditions separately.

1. Solid-wall boundaries. Since the normal velocity vanishes at the wall we have $\int_{\partial\Omega} q'^* R A_n q' ds = 0$, and the matrix $A_n$ has only one negative eigenvalue at the boundary. The well-posedness of the problem then follows directly (also without linearization) for the boundary condition $u_n = 0$.

2. Open boundaries. From the number of negative eigenvalues of $A_n$ it follows that at an inflow boundary $(u_n < 0)$, two boundary conditions should be given in the subsonic region where $|u_n| \leq c$ $(|u_n + u_n'| \leq (\varphi + \frac{1}{2}\varphi')/c)$, and three conditions in the supersonic region. At an outflow boundary, one boundary condition is required if the flow is subsonic, and no condition should be given if the flow is supersonic.

If the flow is supersonic, the boundary conditions should be such that, for the variational equations, the values of all three of the variables $u_1'$, $u_2'$ and $\varphi'$ are prescribed at inflow points.

For subsonic flow, the inflow boundary conditions should determine the value of two quantities of type $u_\perp' - a_1(u_n' + \varphi'/c)$ and $u_n' - \varphi'/c - a_2(u_n' + \varphi'/c)$ since $u_\perp'$ and $u_n' - \varphi'/c$ are the characteristic combinations corresponding to the negative eigenvalues of $A_n$. Not all such combinations give well-posed problems, but if $c + u_n + 2u_n a_1^2 - (c - u_n)a_2^2 \geq 0$, the limited growth rate for the energy norm of $\underset{\sim}{q}'$ gives a direct proof of well-posedness. We may, e.g., choose $a_1 = a_2 = 0$. This condition can be achieved by prescribing $u_\perp$ and $u_n - 2\varphi^{1/2}$ at these inflow points, see Elvius and Sundström [9], since then $u_\perp' = 0$ and $u_n' - 2(\varphi + \varphi')^{1/2} + 2\varphi^{1/2}$ $= u_n' - 2\varphi'/(\varphi^{1/2} + (\varphi + \varphi')^{1/2}) \approx u_n' - \varphi_n'/\varphi^{1/2} = 0$. Actually, this boundary condition ensures that only bounded growth can result for all finite disturbances, as long as $\min\{\varphi^{1/2}, (\varphi + \varphi')^{1/2}\} + u_n + u_n'$ is positive. Inserting the complete expressions for $u_\perp'$ and $u_n'$ in the boundary integral, the integrand becomes

$$
\begin{aligned}
&(u_n + u_n')[(\varphi + \varphi')(u_n'^2 + u_\perp'^2) + \varphi'^2] + (2\varphi + \varphi')u_n'\varphi' \\
&\quad = \varphi'^2\{(u_n + u_n')[1 + 4(\varphi + \varphi')((\varphi + \varphi')^{1/2} + \varphi^{1/2})^{-2}] \\
&\qquad\qquad + 2(2\varphi + \varphi')((\varphi + \varphi')^{1/2} + \varphi^{1/2})^{-1}\} \\
&\quad = \varphi'^2((\varphi + \varphi')^{1/2} + \varphi^{1/2})^{-2}\{(u_n + u_n')(6\varphi + 5\varphi' + 2(\varphi + \varphi')^{1/2}\varphi^{1/2}) \\
&\qquad\qquad + 2(2\varphi + \varphi')((\varphi + \varphi')^{1/2} + \varphi^{1/2})\} .
\end{aligned}
$$

If $0 < \varphi < \varphi + \varphi'$, the value of

$$\frac{2(2\varphi + \varphi')((\varphi + \varphi')^{1/2} + \varphi^{1/2})}{6\varphi + 5\varphi' + 2(\varphi + \varphi')^{1/2}\varphi^{1/2}}$$

is always larger than $\varphi^{1/2}$ while if $0 < \varphi + \varphi' < \varphi$, this expression has $(\varphi + \varphi')^{1/2}$ as its lower bound. For $u_n + u_n' + \mathrm{Min}\{\varphi^{1/2}, (\varphi + \varphi')^{1/2}\} \geq 0$, the inflow part of the boundary then provides a nonnegative contribution to the boundary integral.

Another possible inflow boundary condition is to prescribe both $u_n$ and $u_\perp$, so that $a_1 = 0$, $a_2 = -1$. This condition was apparently first suggested by Rousseau [16]. The value of $|a_2|$ is then so large that we cannot use the classical energy method, but as first shown by Elvius and Kreiss (private communication), well-posedness can be proved using Kreiss' normal mode analysis technique.

An alternative boundary condition, discussed in the paper by Elvius and Sundström [9] is to give $u_\perp$ and $\varphi$ at inflow points. This corresponds to the choice $a_1 = 0$, $a_2 = 1$. For problems in only one space dimension this is a well-posed condition. The value of $a_2$ is, however, so large that we cannot use the energy method as above. For the complete two-dimensional problem, Elvius and Sundström did not analyze the well-posedness properties, but numerical experiments indicated that it might actually be an ill-posed set of boundary conditions. This conjecture has been confirmed by a complete analysis by Elvius and Kreiss (private communication).

At outflow parts of the boundary ($u_n > 0$), we should, in the subsonic case, give one boundary condition. For the variational equations this condition should *prescribe the value of a combination of the form $u_n' - \varphi'/c - b_1 u_\perp' - b_2(u_n' + \varphi'/c)$. As before, well-posedness follows directly from the positivity of the integrand $q'^* RA_n q'$ in the growth equation if

$$(c + u_n)(u_n' + \varphi'/c)^2 + 2u_n u_\perp'^2 - (c - u_n)(b_1 u_\perp' + b_2(u_n' + \varphi'/c))^2 \geq 0$$

for all $u'$ and $u_n' + \varphi'/c$. The simplest choices of the parameters $b_1, b_2$ satisfying this condition are: 1) $b_1 = b_2 = 0$, i.e., giving $u_n' - \varphi'/c$, which may be achieved by specifying $u_n - 2\varphi^{1/2}$; 2) $b_1 = 0$, $b_2 = -1$, i.e., giving $u_n'$; and 3) $b_1 = 0$, $b_2 = 1$, i.e., specifying $\varphi$ also yields a well-posed problem.

All of these conditions guarantee bounded growth for finite disturbances. In each case, the integrand $q'^* RA_n(q + q')q'$ is strictly positive as long as $u_n + u_n' > 0$.

For the open boundary problems we have several possible sets of boundary conditions which all satisfy the necessary and sufficient well-posedness conditions. If the problem is part of a telescoping technique or nested integration, or if any arbitrary type of boundary data can be obtained from measurements, the choice between these different possibilities may be difficult.

The experiments reported by Elvius and Sündstrom [9] do not show any large differences between the results from a numerical model, using either boundary conditions giving the value of the combination $u_n' - \varphi'/c$ and $u_\perp'$ at inflow and $u_n' - \varphi'/c$ at outflow or $u_n'$ and $u_\perp'$ at inflow and $u_n'$ at outflow. Their results examined a long-wave solution with small long-wave or short-wave disturbances.

Further experiments by Elvius (private communication) indicate that for solutions with a less pronounced long-wave character, the first alternative is less susceptible to boundary disturbances which may arise when the normal velocity is small and changes sign.

In one of the first papers on limited-area integration of the shallow-water equations, Charney [2] suggested a quite different set of boundary conditions. Since it is both inefficient and difficult to implement, this set is now primarily of historical interest. It is still worth analyzing, since it illustrates the hazards of intuitive deductions. At that time, most experimental and routine work on numerical weather prediction was done with "balanced" forecast models. The simplest version, the barotropic vorticity equation, can be considered as a further simplification of the shallow-water equations. In the derivation of this approximation, system C is first transformed into a set of three *prognostic equations for the divergence $D = \nabla \cdot \underset{\sim}{u}$, the vorticity $\zeta = \partial u_2/\partial x_1 - \partial u_1/\partial x_2$, and $\varphi$, respectively by differentiating the equations of motion with respect to x and y and combining the results. This differentiated system is then simplified by using a steady-state approximation in the first and third of these equations and by keeping only the nondivergent advection terms in the vorticity equation. By this approximation, the whole system is condensed into one prognostic equation in one dependent variable, the stream function $\psi$. For this vorticity equation, Charney, Fjørtoft, and von Neumann [3] concluded that two boundary conditions should be given at inflow points of an open boundary and one condition at outflow points. They suggested that the stream function (and thus the normal component of the velocity) should be specified at all boundary points, and in addition, the vorticity at inflow parts of the boundary. It is, easy to show that their conclusion on the number of boundary conditions was correct and that the suggested conditions make the problem well-posed, cf. Sundström [19].

Charney's proposed boundary conditions for the shallow-water equationswereapparently based on the idea that since the number of boundary conditions is the same for the vorticity equation and the shallow water equations, the type of conditions should be similar. He therefore suggested that $u_n$ should be prescribed at all boundary points, and as the second quantity to be given at inflow parts of the boundary he chose the "potential vorticity"

$$P = \frac{\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} + f}{\varphi}$$

If the differentiated version of the shallow-water equations is formulated in terms of P, D, and $\varphi$, one of the three equations is $(d/dt)P = 0$. This differentiated system requires three boundary conditions at inflow points, one more than the number required for the single vorticity equation, and one of these conditions may be the prescription of P. However, this is not a valid argument for the usefulness of the potential vorticity as a boundary condition for the undifferentiated system. We cannot just pick two of the three necessary inflow conditions for the differentiated system and expect them to form an appropriate set of boundary conditions for the shallow-water equations.

The danger of using the boundary conditions that Charney suggested can be shown directly. For the variational equations, they give $u'_n = 0$ and $(\partial/\partial n)u'_\perp - P\varphi' = 0$ if $u_n < 0$. The last condition cannot be used directly, but it can be combined with the prognostic equation for $u'_\perp$ to yield an equation for the inflow boundary values of $(\partial/\partial t)u'_\perp$ which involves only boundary quantities. The tangential velocity is then determined by integrating this equation from $t = 0$.

This is a very complicated way of computing the inflow values of $u'_\perp$. Additionally, this approach has the liability that a small error committed initially, or at any later time $t_0$, will influence the boundary values at all later times. These errors will spread into the region of integration contaminating the solution.

## 5. Effects of Viscous Terms

As described in the introduction, small viscous terms are often added to the systems of equations we are considering. They are often introduced to provide a dissipative filter for a numerical approximation. In other cases there is a physical motivation for using viscous terms to represent diffusive transport (eddy flux) of momentum and heat. Both the viscosity and heat conduction coefficients' are usually very small, but since these terms change the character of the differential equations, we have to reinvestigate the boundary conditions.

For system A (the Eulerian equations) the general viscous form of (2.1) is the compressible Navier-Stokes equations, here written in the special form

$$\frac{d}{dt} u + \alpha\nabla p + F = \alpha[\mu\nabla^2 u + (\lambda + \mu)\nabla(\nabla\cdot u)]$$

(5.1) $$\frac{d}{dt} \alpha - \alpha\nabla\cdot u = K_H p^{-1/\gamma}\nabla^2(\alpha p^{1/\gamma})$$

$$\frac{d}{dt} p + p\gamma\nabla\cdot u = 0$$

where $\mu$ and $\lambda$ are the Lame/constants, and where the heat exchange term represents eddy flux of potential temperature. Using the potential temperature $\theta = (p_\perp^{1-1/\gamma}/R)\ \alpha p^{1/\gamma}$ as dependent variable instead of $\alpha$, the second equation may then be simplified to

$$\frac{d}{dt}\theta = K_H\nabla^2\theta$$

Here, we shall only study the simple viscous form obtained when $\mu = -\lambda = \nu\alpha^{-1}$. In vector notation, we have

$$\frac{\partial}{\partial t} q + \sum_{j=1}^{3} A_j(q) \frac{\partial}{\partial x_j} q + F = B\nabla^2 q$$

with $q = (u_1, u_2, u_3, \theta, p)^t$,

$$A_1 = \begin{pmatrix} u_1 & 0 & 0 & 0 & \alpha \\ 0 & u_1 & c & 0 & 0 \\ 0 & 0 & u_1 & 0 & 0 \\ 0 & 0 & 0 & u_1 & 0 \\ p\gamma & 0 & 0 & 0 & u_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} u_2 & 0 & 0 & 0 & 0 \\ 0 & u_2 & 0 & 0 & \alpha \\ 0 & 0 & u_2 & 0 & 0 \\ 0 & 0 & 0 & u_2 & 0 \\ 0 & p\gamma & 0 & 0 & u_2 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} u_3 & 0 & 0 & 0 & 0 \\ 0 & u_3 & 0 & 0 & 0 \\ 0 & 0 & u_3 & 0 & a \\ 0 & 0 & 0 & u_3 & 0 \\ 0 & 0 & p\gamma & 0 & u_3 \end{pmatrix} \quad C = \begin{pmatrix} 0 & -f & 0 & 0 & 0 \\ f & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$B = \alpha \begin{pmatrix} \mu & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This is an incompletely parabolic system, see Strikwerda [18]. As before, the matrices $A_1$, $A_2$, $A_3$, and B can be simultaneously symmetrized by multiplication from the left by

$$R = \alpha^{-1}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & a^2\,\frac{p\gamma\alpha}{\theta^2} & 0 \\ 0 & 0 & 0 & 0 & \frac{a}{p\gamma} \end{pmatrix}$$

where $a^2$ is the same positive parameter which occurred in the transformation matrix R in section 2. If the solution (and then also R and the coefficient matrices) are slowly varying in space and time, we can use the general results of Strikwerda [18] from which it follows that, for well-posedness, it is necessary and sufficient that the family of linearized variational equations

$$\frac{\partial}{\partial t}\underset{\sim}{q}' + \sum_{j=1}^{3} A_j(\underset{\sim}{q})\,\frac{\partial}{\partial x_j}\underset{\sim}{q}' + \underset{\sim}{F}' = B\nabla^2\underset{\sim}{q}'$$

formawell-posed problem.

We first study the rigid wall problem. A normal-mode analysis shows that four boundary conditions are required. One of these conditions, $u_n' = 0$, follows immediately from the solid-wall condition $u_n = 0$. The three remaining conditions may be chosen as $a_1 u_{\perp 1}' + (1-a_1)\mu\frac{\partial}{\partial n}u_{\perp 1}' = 0$, $a_2 u_{\perp 2}' + (1-a_2)\mu\frac{\partial}{\partial n}u_{\perp 2}' = 0$, and $a_3\theta' + (1-a_3)K_H\frac{\partial}{\partial n}\theta' = 0$ for some nonnegative $a_1$, $a_2$, and $a_3$.

If $a_1 = a_2 = a_3 = 1$, these conditions correspond to a "nonslip", perfectly conducting wall, while if $a_1 = a_2 = a_3 = 0$, they represent a "perfect slip", thermally insulated wall. The well-posedness of the solid-wall *problem with these boundary conditions can be demonstrated by the energy method.

For small values of $\mu$ and $K_H$, we can expect boundary layers of thickness $\mathcal{O}(\mu^{1/2})$ and $\mathcal{O}(K_H^{1/2})$ at the solid wall. If some $a_i \neq 0$, the value of the corresponding variable can be expected to change by a finite amount within the boundary layer. In such cases, we must use a numerical approximation that resolves the boundary layer satisfactorily. If all $a_i = 0$, the variables will only change by an amount proportional to the boundary layer thickness, and the difference between the solutions to the viscous and inviscid equations is then always small for small $\mu$ and $K_H$.

In problems with open boundaries, there are no obvious physical boundary conditions. As before, we have some freedom in choosing the most suitable mathematical boundary conditions. For the viscous equations, it is not sufficient to 'prescribe a set of conditions that makes the problem well-posed. Since we know that there should be no boundary layers at the inflow and outflow boundaries, we have to choose the mathematical boundary conditions accordingly. This problem has been studied by Gustafsson and Sundström [11]. 'They showed that for $\mu, K_H \to 0$, the conditions must yield a well-posed set of boundary conditions for the inviscid equations, i.e., singular boundary layers should not occur.

At an inflow boundary, the normal-mode analysis shows that five boundary conditions should be prescribed while only four were appropriate for the inviscid equations. Using the energy method, we can show that the following set of boundary conditions gives a well-posed problem with no essential boundary layer:

$$u'_n - p'(\alpha/p\gamma)^{1/2} = 0, \quad u'_{\perp 1} = u'_{\perp 2} = \theta' = 0, \text{ and } \mu \frac{\partial u'_n}{\partial n} = 0.$$

A similar set of conditions consistent with the other type of inviscid conditions $(u'_n = u'_{\perp 1} = u'_{\perp 2} = \theta' = 0)$ is less trivial to find. This is because none of these inviscid conditions include the pressure disturbance $p'$.

At an outflow boundary, the viscous equations require four boundary conditions and the inviscid problem only one. If we prefer to give $u'_n = 0$ as the inviscid condition, we can now prescribe

$$u'_n = 0, \quad \mu \frac{\partial}{\partial n} u'_{\perp 1} = \mu \frac{\partial}{\partial n} u'_{\perp 2} = K_H \frac{\partial}{\partial n} \theta' = 0$$

The inviscid condition $u'_n - p'(\alpha/p\gamma)^{1/2} = 0$ can also be extended to the viscous case by giving

$$u'_n - p'_\perp(\alpha/p\gamma)^{1/2} + \mu(\alpha/p\gamma)^{1/2} \frac{\partial u'_n}{\partial n} = 0, \quad \mu \frac{\partial}{\partial n} u'_{\perp 1} = \mu \frac{\partial}{\partial n} u'_{\perp 2} = K_H \frac{\partial}{\partial n} \theta' = 0.$$

In both these cases, we can use the energy method to prove well-posedness, and the absence of boundary layers as $\mu, K_H \to 0$ follows directly.

All these boundary conditions may be expressed in terms of $\alpha'$ and $p'$ instead of $\theta'$ by using the identity

$$\theta' = \theta \left( \frac{\alpha'}{\alpha} + \frac{p'}{p\gamma} \right)$$

which is valid for small perturbations.

For the hydrostatic approximation (B1) to these Eulerian equations, the analysis is at least not simplified by including viscous terms. Even if we could find a set of boundary conditions that make the linearized variational equations with constant coefficients well-posed, we would not know if this is, in some sense, true for the complete equations.

For the anelastic equations (B2), viscous terms may easily be included. To find the 'proper number of boundary conditions, we may as before study only the linearized variational equations with constant coefficients. The result of the analysis is that at all boundaries, four boundary conditions should be prescribed. At an inflow boundary this is the same number as required by the inviscid equations. We may use the same conditions as before, $u'_n = u'_{\perp 1} = u'_{\perp 2} = \theta' = 0$ for the linearized, constant coefficient equations.

At an outflow boundary, the inviscid equations required one boundary condition, $u'_n = 0$. This may now be supplemented by

$$\mu \frac{\partial}{\partial n} u'_{\perp 1} = \mu \frac{\partial}{\partial n} u'_{\perp 2} = K_H \frac{\partial}{\partial n} \theta' = 0 .$$

For the solid-wall case, we may either choose the inflow or outflow type of conditions. The choice depends on whether we have nonslip or perfect slip and a perfectly conducting or a thermally insulated wall. The well-posedness of these conditions is easily demonstrated.

For the shallow-water equations (C), we have the same type of behavior as with the Eulerian equations (A). At a solid wall, the obvious condition $u'_n = 0$ must now be supplemented by one more boundary condition, $a_1 u'_1 + (1-a_1)\mu \frac{\partial}{\partial n} u'_1 = 0$. The effects caused by choosing al = 1 and $a_1 = 0$ are similar to the results for system (A).

At an inflow boundary, the boundary condition $u'_n - \varphi'/c = 0$, $u'_\perp = 0$ is easily modified by adding the required third condition $\mu \frac{\partial}{\partial n} u'_n = 0$. If the inviscid conditions are $u'_n = 0$, $u'_\perp = 0$, we cannot find a suitable viscous form by our energy method analysis.

At outflow boundaries, we need two conditions which can be chosen to be $u'_n = 0$ and $\mu \frac{\partial}{\partial n} u' = 0$ or

$$u'_n - \frac{\varphi'}{c} + \frac{\mu}{c} \frac{\partial u'_n}{\partial n} = 0 \quad \text{and} \quad \mu \frac{\partial}{\partial n} u' = 0 ,$$

depending on what type of condition is preferred for the inviscid equations. Well-posedness is easily proven by the energy method, and no artificial boundary layers are generated.

Implications for numerical methods

The analysis of various initial-boundary value problems contained in this paper was motivated by difficulties arising in computational models of the problems we have discussed. It is appropriate to comment on the implications these results have on these numerical models.

'The stability analysis and related error estimates for the approximate methods are the discrete analogues of our well-posedness analysis and estimates of the form (1.3). For a given well-posed problem of the types discussed here, we can always find stable difference approximations and numerical boundary conditions. Examples are given in Gustafsson, Kreiss and Sundström [10] and Elvius and Sundström [9]. Conversely, an approximation cannot have a norm which behaves reasonably if it accurately approximates an ill-posed problem.

When, for a given problem, the number of boundary conditions is overspecified, the difference approximation may well be stable. However, the effective boundary conditions which influence the solution are, in general, difficult to determine, especially for problems in several space dimensions. They may well be a complicated function of the conditions given and bear little resemblance to them.

An additional complication induced by overspecification is that the underlying solution being approximated is not generally continuous. The phenomena associated with approximations to discontinuous solutions have been studied by several authors, a good discussion and summary of these results can be found in section 10 of Thomée [21]. These results may be summarized as follows. If a non-dissipative approximation is used, then high frequency waves emanate from the region of the discontinuity and travel across the domain without losing appreciable amplitude. They will usually travel with the highest fundamental wave speeds and rapidly cover the domain with error. If scalar equations are being approximated, then this region of error can be restricted to the vicinity of the discontinuity by using dissipative approximations. However, these results do not apply to systems of equations as we have here. The errors can propagate away from the discontinuity through other components of the solution. Boundary value overspecification may be regarded as a stationary source of such discontinuities.

In order to avoid the problems associated with the proper selection of boundary conditions, the order and type of the differential equations is often raised to obtain a problem that is easier to analyze and approximate. The equations are usually modified by adding dissipative terms so that the number of boundary conditions is appropriate. Unfortunately, this idea seldom works. If a spurious boundary layer of appreciable size results, the effects are not unlike those discussed above for discontinuities and, unless the dissipative terms are very large, the error introduced at the boundary will again propagate into the interior.

If the boundary conditions are underspecified there are no *a priori* estimates for the differential equations. In order for an approximation to be computable there must be a sufficient number of boundary conditions specified for the approximation. This cannot be fewer than the number required for the differential equation. Additional conditions are usually constructed by means of extrapolations. For an underspecified problem the extrapolation of quantities that should be prescribed results in an unstable or inconsistent approximation of the wrong differential equation.

## REFERENCES


[1] N. Campbell, "Stability analysis of a difference scheme for the Navier-Stokes equations," Numer. Math. 14 (1970), pp. 435-447.

[2] J. Charney, "Integration of the primitive and balance equations," Proceedings of the International Symposium on Numerical Weather Prediction, Tokyo (1962), pp. 131-152.

[3] J. Charney, R. Fjørtoft, and J. von Neumann, "Numerical integration of the barotropic vorticity equation," Tellus 2 (1950), pp. 237-254.

[4] R. Courant and D. Hilbert, Methods of Mathematical Physics, Vol. II, Interscience Publishers, New York (1962).

[5] H. Davies, "On the lateral boundary conditions for the primitive equations," J. Atmos. Sci. 30 (1973), pp. 147-150.

[6] H. Davies, "Reply" to "Comments 'On the lateral boundary conditions for the primitive equations'," J. Atmos. Sci. 31 (1974), pp. 596-597.

[7] E. de Rivas, "Comments 'On the lateral boundary conditions for the primitive equations'," J. Atmos. Sci. 31 (1974), p.596.

[8] J. Dutton, "The nonlinear quasi-geostrophic equation: existence and uniqueness of solutions on a bounded domain," J. Atmos. Sci. 31 (1974), pp. 422-433.

[9] T. Elvius and A. Sundström, "Computationally efficient schemes and boundary conditions for a fine-mesh barotropic model based on the shallow-water equations," Tellus 25 (1973), pp. 132-156.

[10] B. Gustafsson, H.-O. Kreiss and Arne Sundstrom, "Stability theory of difference approximations for mixed initial boundary value problems, II," Math. Comp. 26 (1972), pp. 649-686.

[11] B. Gustafsson and A. Sundström, "Incompletely parabolic systems in fluid dynamics," to appear.

[12] H.-O. Kreiss, "Initial boundary value problems for hyperbolic equations," Comm. Pure and Appl. Math. 23 (1970), pp. 277-298.

[13] H.-O. Kreiss, "Initial boundary value problems for hyperbolic equations," Conference on the Numerical Solution of Differential Equations, ed. A. Dold and B. Eckman, Lecture Notes in Mathematics, No. 363, Springer-Verlag, Berlin (1974).

[14] A. Majds and S. Osher, "Initial-boundary value problems for hyperbolic equations with uniformly characteristic boundaries," Comm. Pure and Appl. Math. 28 (1975), pp. 607-675.

[15] J. Oliger and A. Sundström, "The initial-boundary value problem for the inviscid Eulerian equations for fluid dynamics," to appear.

[16] D. Rousseau, "Experience préliminaire a l'élaboration d'un modèle numérique de prévision régionale," Notes de l'établissement d'etudes et de recherches meteorologiques, 259 (1968).

[17] J. Serrin, "On the uniqueness of compressible fluid motions," Arch. Rat. Mech. Anal. 3 (1959), pp. 271-288.

[18] J. Strikwerda, "Initial boundary value problems for incompletely parabolic systems,' Ph.D. Thesis, Dept. of Math., Stanford University (1976).

[19] A. Sundström, "Stability theorems for the barotropic vorticity equation," Mon. Wea. Rev. 97 (1969), pp. 340-345.

[20] M. Taylor, Pseudo Differential Operators, Lecture Notes in Mathematics, No. 416, Springer-Verlag, Berlin (1974).

[21] V. Thomée, "Stability theory for partial difference operators," SIAM Rev. 11 (1969), pp. 152-195.

[22] A. Wiin-Nielsen, "On the propagation of gravity waves in a hydrostatic, compressible fluid with vertical wind shear," Tellus 17 (1965), pp. 306-320.