FINDING A MAXIMUM INDEPENDENT SET

by

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Abstract

We present an algorithm which finds a maximum independent set in an n-vertex graph in $O(2^{n/3})$ time. The algorithm can thus handle graphs roughly three times as large as could be analyzed using a naive algorithm.

Keywords: algorithm, clique, computational complexity, graph, maximum independent set, NP-complete problem.

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1. Introduction.

A graph $G = (V,E)$ is an ordered pair consisting of a finite set V of vertices and a set of unordered pairs (v,w) of distinct vertices, called edges. Two vertices v,w are adjacent if $(v,w) \in E$. A set S of vertices is independent (or internally stable) if $(v,w) \notin E$ for all $v,w \in S$. A set S of vertices is a clique if $(v,w) \in E$ for all $v,w \in S$. The complement of a graph' $G = (V,E)$ is the graph $\bar{G} = (V,\bar{E})$ where $\bar{E} = \{(v,w) \mid v,w \in V, v \neq w, \text{ and } (v,w) \notin E\}$. Clearly $S \subseteq V$ is an independent set of G if and only if S is a clique of $\bar{G}$. (Note: some authors require that a clique be a maximal set of pairwise adjacent vertices; we do not.)

A path from vl to v_k in a graph $G = (V,E)$ is a sequence of vertices $v_1, v_2, \ldots, v_k$ such that $(v_i, v_{i+1}) \in E$ for $1 \leq i < k$. A set of vertices S is connected if, for all $v,w \in S$, there is a path from v to w containing only vertices in S. The vertices of a graph G can be partitioned into maximal connected subsets, called the connected components of G. If $G = (V,E)$ is a graph and S is a set of vertices the graph $G(S) = (S,E(S))$, where $E(S) = \{(v,w) \in E \mid v,w \in S\}$ is called the subgraph of G induced by the vertex set S.

We consider the problem of finding a maximum-size independent set in a given graph $G = (V,E)$; or, equivalently, finding a maximum-size clique in a given graph. This problem has been studied extensively, but no polynomial-time algorithm is known. In fact, the maximum independent set problem is NP-complete [4,7], and thus is unlikely to have a polynomial-time algorithm. Our goal is to provide an algorithm, which, though not polynomial, is significantly faster in the worst case than the obvious enumeration algorithm or any other algorithm known to us.

Let $n = |V|$. The number of subsets of V is 2^n . By listing each possible subset of V and testing it for independence, one can find a maximum clique in $O(p(n)2^n)$ time, where $p(n)$ is some polynomial. Other algorithms have been proposed [2,9,10], but for none except the one in [10] has a worst-case time bound better than $O(2^n)$ been proved.

We extend the algorithm of [10] to provide an $O(2^{n/3})$ -time algorithm. The algorithm is recursive and depends upon a somewhat complicated case analysis. Though the algorithm is tedious to state in detail, it would be straightforward to program, and we suspect that it would perform well in practice. Nevertheless, its main interest seems to be theoretical; its existence shows that at least one NP-complete problem can be solved in a time boundsignificantly better than that of the obvious enumeration algorithm. For a similar algorithm to solve another NP-complete problem, see [5].

It is also worth noting that the maximum number of independent sets maximal with respect to the subset relation in a graph of n vertices is $3^{n/3}$. One could find a maximum-size independent set by enumerating all maximal independent sets (using an algorithm such as in [1, 3,6,8]) and choosing the largest. However, the algorithm to be proposed is substantially better than even this method, in the worst case.

The algorithm uses a recursive, or backtracking scheme. Its starting point is the following observation. Let $v \in V$. Let $A(v)$ be the set of vertices adjacent to v . Then any maximum independent set either contains v or it does not. Thus any maximum independent set of G is either $\{v\}$ combined with a maximum independent set in $G(V - \{v\} - A(v))$, or it is a maximum independent set in $G(V - \{v\})$.

We extend this idea. For any $S \subseteq V$, let $A(S) = \bigcup_{v \in S} A(v)$. If $S \subseteq V$, then any maximum independent set I in G consists of an independent Set $I \cap S$ in G(S) and a maximum independent set I-S in $G(V-S-A(I))$. Our algorithm selects a subset $S \subseteq V$, finds each independent set J in G(S) , and, for each such J , recursively finds a maximum independent set in G(V-S-A(J)) .

We improve this method further by introducing the concept of dominance. If $S \subseteq V$ and I, J are independent in G(S) , we say I dominates J if, for any $J' \subseteq V-S$ such that $J \cup J'$ is independent, there is a set $I' \subseteq V-S$ such that $I \cup I'$ is independent and $|I \cup I'| \geq |J \cup J'|$. For any such dominated J , we need not solve a subproblem, since we get an independent set at least as large by solving a subproblem for I .

Dominance is important because in certain cases it can be confirmed quickly. We give two examples which are used extensively in the algorithm. Let $v \in V$. Let $S = \{v\} \cup A(v)$. If $w \in A(v)$, then $\{v\}$ dominates $\{w\}$ in S , since $I \subseteq V-S$ and $I \cup \{w\}$ independent implies $I \cup \{v\}$ independent. Similarly, $\{v\}$ dominates $\emptyset$ in S .

Let $S \subseteq V$. Let I and $J = I \cup \{v\}$ be independent in G(S) . Suppose $(V-S) \cap A(v) = \{w_1, w_2\}$. In $S \cup \{w_1, w_2\}$, J dominates both $I \cup \{w_1\}$ and $I \cup \{w_2\}$. We distinguish three possibilities.

(i) $(w_1, w_2) \in E$ or $I \cap A(\{w_1, w_2\}) \neq \emptyset$. Then J dominates I in S : if $I' \subseteq V-S$ and $I' \cup I$ is independent, then $|I' \cap \{w_1, w_2\}| \leq 1$. Thus $J' = I' - \{w_1, w_2\}$ satisfies $|J' \cup J| \geq |I' \cup I|$ and $J' \cup J$ is independent.

(ii) $(w_1,w_2) \notin E$, $I \cap A(\{w_1,w_2\}) = \emptyset$, and $|(V - S - A(J)) \cap A(\{w_1,w_2\})| \leq 1$.
Then I dominates J in S (and $I \cup \{w_1,w_2\}$ dominates J in $S \cup \{w_1,w_2\}$) : If $J' \subseteq V-S$ and $J' \cup J$ is independent, then $I' = (J' \cup \{w_1,w_2\}) - A(\{w_1,w_2\})$ satisfies $|I' \cup I| \geq |J' \cup J|$ and $I' \cup I$ is independent.

(iii) $(w_1,w_2) \notin E$, $I \cap A(\{w_1,w_2\}) = \emptyset$, and $|(V - S - A(J)) \cap A(\{w_1,w_2\})| \geq 2$.
In this case we need further information to determine whether I dominates J or vice-versa.

In summary, the algorithm selects a set $S \subseteq V$, determines a set of dominating independent sets in S using the two observations above, and recursively solves one subproblem for each dominating set.

2. The Algorithm.

A detailed specification of the algorithm appears below. A call maxset(S) will return an integer which is the size of a maximum independent set in $G(S)$; the graph $G = (V,E)$ is assumed to be a global variable. The statement of the algorithm consists of a sequence of cases and subcases. The first case which applies is used to define the value of maxset(S) . Thus, inside a given case, the hypotheses of all previous cases can be assumed to be false. It is easy to modify the algorithm so that it returns a maximum independent set as well as the size of such a set.

procedure maxset (V);

begin

0: V is not connected.

Let $V_1, V_2, \ldots, V_k$ be the connected components of V .

Note that every maximum independent set consists of a union of maximum independent sets, one from each connected component.

Let maxset = $\sum_{i=1}^{k}$ maxset(V_i) .

V is connected. Let v be a vertex of minimum degree.

1: $d(v) = 1$.

Let $A(v) = \{w\}$.

Let maxset = 1 + maxset$(V - \{v,w\})$.

2: $d(v) = 2$.

2.1: $d(w) = 2$ for all $w \in V$.

Note that the vertices of V form a cycle.

Let maxset = $\lfloor |V|/2 \rfloor$.

There exist v,w such that d(v) = 2 , $d(w_1) \geq 3$, and $(v,w_1) \in E$. Let $A(v) = \{w_1,w_2\}$.

2.2: $(w_1,w_2) \in E$.

Let $\underline{\text{maxset}} = 1 + \underline{\text{maxset}}(V - \{v,w_1,w_2\})$.

2.3: $(w_1,w_2) \notin E$.

Let $\underline{\text{maxset}} = \max\{1 + \underline{\text{maxset}}(V - \{v,w_1,w_2\})$,

$2 + \underline{\text{maxset}}(V - A(w_1) - A(w_2))\}$.

3: d(v) = 3.

Let $A(v) = \{w_1,w_2,w_3\}$.

3.1: $(w_1,w_2), (w_1,w_3), (w_2,w_3) \in E$.

Let $\underline{\text{maxset}} = 1 + \underline{\text{maxset}}(V - \{v,w_1,w_2,w_3\})$.

3.2: $(w_1,w_2),(w_1,w_3) \in E$ (or any symmetric case).

Let $\underline{\text{maxset}} = \max\{1 + \underline{\text{maxset}}(V - \{v,w_1,w_2,w_3\})$,

$2 + \underline{\text{maxset}}(V - A(w_2) - A(w_3))\}$.

3.3: $(w_1,w_2) \in E$ (or any symmetric case).

For i = 1,2,3 , let $\bar{A}_i = V - \{w_1,w_2,w_3\} - A(w_i)$.

Note that $|\bar{A}_1|, |\bar{A}_2| \leq |V|-5$, $|\bar{A}_3| \leq |V|-6$.

3.3.1: $|\bar{A}_1 \cap \bar{A}_3| \leq |\bar{A}_2 \cap \bar{A}_3| = |V|-6$ (or the symmetric case).

Note that $\bar{A}_2 \cap \bar{A}_3 = \bar{A}_3$. Thus $\{w_2,w_3\}$ dominates $\{w_1,w_3\}$.

Let $\underline{\text{maxset}} = \max\{1 + \underline{\text{maxset}}(V - \{v,w_1,w_2,w_3\})$,

$2 + \underline{\text{maxset}}(\bar{A}_3)\}$.

3.3.2: $|\bar{A}_1 \cap \bar{A}_3|, |\bar{A}_2 \cap \bar{A}_3| \le |V|-7$.

Let $\underline{\text{maxset}} = \max\{1 + \text{maxset}(V - \{v, w_1, w_2, w_3\})$,

$2 + \underline{\text{maxset}}(\bar{A}_1 \cap \bar{A}_3)$,

$2 + \underline{\text{maxset}}(\bar{A}_2 \cap \bar{A}_3)\}$.

3.4: $(w_i, w_j) \notin E$ for $i,j \in \{1,2,3\}$.

For $i = 1,2,3$, let $\bar{A}_i = V - \{w_1, w_2, w_3\} - A(w_i)$.

Note that $|\bar{A}_i| \le |V|-6$ for $i = 1,2,3$.

3.4.1: $|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| \ge |V|-7$.

Set $\{w_1, w_2, w_3\}$ dominates $\{w_i, w_j\}$ for $i,j \in \{1,2,3\}$.

Let $\underline{\text{maxset}} = \max\{1 + \underline{\text{maxset}}(V - \{v, w_1, w_2, w_3\})$,

$3 + \underline{\text{maxset}}(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3)\}$.

3.4.2: $|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| = |V|-8$ or $|V|-9$.

If, for some i,j , $|\bar{A}_i \cap \bar{A}_j| \le |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3|+1$, then $\{v_i, w_j\}$ is dominated by $\{w_1, w_2, w_3\}$.

For distinct i, j , k ,

$$|\bar{A}_i| = |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| + |\bar{A}_i \cap \bar{A}_j - (\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3)| + |\bar{A}_i \cap \bar{A}_k - (\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3)| ,$$

$|\bar{A}_i| \le |V|-6$, and $|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| \ge |V|-9$.

Thus $|\bar{A}_i \cap \bar{A}_j| > |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3|+2$ for only one possible pair $i \ne j$. Let 1, 2 be the pair (if any).

3.4.2.1: $|\bar{A}_i \cap \bar{A}_j| \leq |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| + 1$ for all $i \neq j$.

Let maxset = $\max\{1 + \text{maxset}(V - \{v, w_1, w_2, w_3\})$,
$3 + \text{maxset}(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3)\}$.

3.4.2.2: $|\bar{A}_1 \cap \bar{A}_2| \geq |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| + 2$ (or any symmetric case).

Let maxset = $\max\{1 + \text{maxset}(V - \{v, w_1, w_2, w_3\})$,
$2 + \text{maxset}(\bar{A}_1 \cap \bar{A}_2)$,
$3 + \text{maxset}(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3)\}$.

3.4.3: $|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| \leq |V| - 10$.

3.4.3.1: $|\bar{A}_i \cap \bar{A}_j| \leq |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| + 1$ for all $i \neq j$.
Same as 3.4.2.1.

3.4.3.2: $|\bar{A}_1 \cap \bar{A}_2| \geq |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| + 2$ (or any symmetric case).

Same as 3.4.2.2.

3.4.3.3: $|\bar{A}_1 \cap \bar{A}_2|, |\bar{A}_1 \cap \bar{A}_3| \geq |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| + 2$ (or any symmetric case).

Let maxset = $\max\{1 + \text{maxset}(V - \{v, w_1, w_2, w_3\})$,
$2 + \text{maxset}(\bar{A}_1 \cap \bar{A}_2)$,
$2 + \text{maxset}(\bar{A}_1 \cap \bar{A}_3)$,
$3 + \text{maxset}(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3)\}$.

3.4.3.4: $|\bar{A}_1 \cap \bar{A}_2|, |\bar{A}_1 \cap \bar{A}_3|, |\bar{A}_2 \cap \bar{A}_3| \geq |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| + 2$.

For $i = 1,2,3$, let $u_{i1}, u_{i2} \in (\bar{A}_j \cap \bar{A}_k) - \bar{A}_i$ $(j,k \neq i)$.

3.4.3.4.1: $|\bar{A}_j \cap \bar{A}_k| = |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| + 2$ and $(u_{i1}, u_{i2}) \in E$ for some distinct i, j, k.

Then $\{w_1, w_2, w_3\}$ dominates $\{w_j, w_k\}$. Same as 3.4.3.3.

3.4.3.4.2: $|\bar{A}_j \cap \bar{A}_k| = |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| + 2$ and $(u_{i1}, u_{i2}) \notin E$ for all distinct i, j, k. Let

$$\begin{aligned}\underline{\text{maxset}} = \max\{&1 + \underline{\text{maxset}}(V - \{v, w_1, w_2, w_3\}), \\ &4 + \underline{\text{maxset}}(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 - A(u_{11}) - A(u_{12})), \\ &4 + \underline{\text{maxset}}(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 - A(u_{21}) - A(u_{22})), \\ &4 + \underline{\text{maxset}}(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 - A(u_{31}) - A(u_{32})), \\ &3 + \underline{\text{maxset}}(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3)\}.\end{aligned}$$

3.4.3.4.3: $|\bar{A}_1 \cap \bar{A}_2|, |\bar{A}_1 \cap \bar{A}_3| = |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| + 2$ (or any symmetric case).

Let

$$\begin{aligned}\underline{\text{maxset}} = \max\{&1 + \underline{\text{maxset}}(V - \{v, w_1, w_2, w_3\}), \\ &4 + \underline{\text{maxset}}(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 - A(u_{31}) - A(u_{32})), \\ &4 + \underline{\text{maxset}}(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 - A(u_{21}) - A(u_{22})), \\ &2 + \underline{\text{maxset}}(\bar{A}_2 \cap \bar{A}_3), \\ &3 + \underline{\text{maxset}}(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3)\}.\end{aligned}$$

3.4.3.4.4: $|\bar{A}_1 \cap \bar{A}_2| = |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| + 2$ (or any symmetric case).

Let

$$\underline{\text{maxset}} = \max\{1 + \underline{\text{maxset}}(V - \{v, w_1, w_2, w_3\}),$$
$$4 + \underline{\text{maxset}}(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 - A(u_{31}) - A(u_{32})),$$
$$2 + \underline{\text{maxset}}(\bar{A}_1 \cap \bar{A}_3),$$
$$2 + \underline{\text{maxset}}(\bar{A}_2 \cap \bar{A}_3),$$
$$3 + \underline{\text{maxset}}(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3)\} .$$

3.4.3.4.5: $|\bar{A}_i \cap \bar{A}_j| \geq |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| + 3$ for $i \neq j$.

Let

$$\underline{\text{maxset}} = \max\{1 + \underline{\text{maxset}}(V - \{v, w_1, w_2, w_3\}),$$
$$2 + \underline{\text{maxset}}(\bar{A}_1 \cap \bar{A}_2),$$
$$2 + \underline{\text{maxset}}(\bar{A}_1 \cap \bar{A}_3),$$
$$2 + \underline{\text{maxset}}(\bar{A}_2 \cap \bar{A}_3),$$
$$3 + \underline{\text{maxset}}(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3)\} .$$

4: $d(v) = 4$.

4.1: $d(w) = 4$ for all vertices w .

4.1.1: There are vertices v, w such that $(v,w) \notin E$ and $|A(v) \cap A(w)| \geq 2$.

4.1.1.1: $|A(v) \cap A(w)| \geq 3$.

Then $\{v,w\}$ dominates both $\{v\}$ and $\{w\}$ in $\{v,w\}$.

Let

$$\underline{\text{maxset}}\{2 + \underline{\text{maxset}}(V - \{v,w\} - A(v) - A(w)),$$
$$\underline{\text{maxset}}(V - \{v,w\})\} .$$

4.1.1.2: $(A(v) \cap A(w)| = 2$.

Let $x,y \in A(v) - A(w)$, $q,r \in A(w) - A(v)$.

Let $\bar{A}(z) = V - \{z\} - A(z)$ for $z \in V$.

4.1.1.2.1: $(x,y),(q,r) \in E$.

Then $\{v,w\}$ dominates both $\{v\}$ and $\{w\}$ in $\{v,w\}$.

Let

$$\underline{\text{maxset}} = \max\{2 + \underline{\text{maxset}}(\bar{A}(v) \cap \bar{A}(w)),$$
$$\underline{\text{maxset}}(V - \{v,w\})\} .$$

4.1.1.2.2: $(x,y) \in E$, $(q,r) \notin E$ (or symmetric ease) .

Let

$$\underline{\text{maxset}} = \max\{2 + \underline{\text{maxset}}(\bar{A}(v) \cap \bar{A}(w)),$$
$$3 + \underline{\text{maxset}}(\bar{A}(v) \cap \bar{A}(w) \cap \bar{A}(q) \cap \bar{A}(r)),$$
$$\underline{\text{maxset}}(V - \{v,w\})\} .$$

4.1.1.2.3: $(x,y),(q,r) \notin E$,

$|\bar{A}(v) \cap \bar{A}(w) \cap \bar{A}(q) \cap \bar{A}(r)| \geq |V| - 9$

(or symmetric case).

Let

$\underline{maxset} = \max\{3 + \underline{maxset}(\bar{A}(v) \cap \bar{A}(w) \cap \bar{A}(x) \cap \bar{A}(y))$,

$3 + \underline{maxset}(\bar{A}(v) \cap \bar{A}(w) \cap \bar{A}(q) \cap \bar{A}(r))$,

$\underline{maxset}(V - \{v,w\})\}$.

4.1.1.2.4: $(x,y), (q,r) \notin E$,

$|\bar{A}(v) \cap \bar{A}(w) \cap \bar{A}(q) \cap \bar{A}(r)|$,

$|\bar{A}(v) \cap \bar{A}(w) \cap \bar{A}(x) \cap \bar{A}(y)| \leq |V| - 10$.

Let

$\underline{maxset} = \max\{2 + \underline{maxset}(\bar{A}(v) \cap \bar{A}(w))$,

$3 + \underline{maxset}(\bar{A}(v) \cap \bar{A}(w) \cap \bar{A}(x) \cap \bar{A}(y))$,

$3 + \underline{maxset}(\bar{A}(v) \cap \bar{A}(w) \cap \bar{A}(q) \cap \bar{A}(r))$,

$\underline{maxset}(V - \{v,w\})\}$.

4.1.2: If $(v,w) \notin E$, then $|A(v) \cap A(w)| \leq 1$.

Let $A(v) = \{w_1, w_2, w_3, w_4\}$. For $i = 1,2,3,4$, let $\bar{A}_i = V - A(v) - A(w_i)$. Then, for $i \neq j$, $\bar{A}_i \cap \bar{A}_j = \emptyset$. Also, if $(w_i, w_j), (w_i, w_k) \in E$, then $(w_j, w_k) \in E$. *

4.1.2.1: $(w_1, w_i) \in E$ for $i = 2,3,4$ (or any symmetric case).

It follows from * above that the problem graph is a complete graph of five vertices. Let $\underline{maxset} = 1$.

4.1.2.2: $(w_1,w_2),(w_1,w_3),(w_2,w_3) \in E$,

$(w_1,w_4),\ (w_2,w_4),\ (w_3,w_4) \notin E$ (or any symmetric case).

Let $\underline{\text{maxset}} = \max\{1+\underline{\text{maxset}}(V-\{v\}-A(v))$,

$2+\underline{\text{maxset}}(\bar{A}_1 \cap \bar{A}_4)$,

$2+\underline{\text{maxset}}(\bar{A}_2 \cap \bar{A}_4)$,

$2+\underline{\text{maxset}}(\bar{A}_3 \cap \bar{A}_4)\}$.

4.1.2.3: $(w_1,w_2),(w_3,w_4) \in E$,

$(w_1,w_3),(w_1,w_4),(w_2,w_3),(w_2,w_4) \notin E$ (or any symmetric case).

Let $\underline{\text{maxset}} = \max\{1+\underline{\text{maxset}}(V-\{v\}-A(v))$,

$2+\underline{\text{maxset}}(\bar{A}_1 \cap \bar{A}_3)$,

$2+\underline{\text{maxset}}(\bar{A}_2 \cap \bar{A}_3)$,

$2+\underline{\text{maxset}}(\bar{A}_1 \cap \bar{A}_4)$,

$2+\underline{\text{maxset}}(\bar{A}_2 \cap \bar{A}_4)\}$.

4.1.2.4: $(w_1,w_2) \in E$,

$(w_1,w_3),\ (w_2,w_3),\ (w_1,w_4),\ (w_2,w_4),\ (w_3,w_4) \notin E$

(or any symmetric case).

Let $\underline{\text{maxset}} = \max\{1+\underline{\text{maxset}}(V-\{v\}-A(v))$,

$2+\underline{\text{maxset}}(\bar{A}_1 \cap \bar{A}_3)$,

$2+\underline{\text{maxset}}(\bar{A}_2 \cap \bar{A}_3)$,

$2+\underline{\text{maxset}}(\bar{A}_1 \cap \bar{A}_4)$,

$2+\underline{\text{maxset}}(\bar{A}_2 \cap \bar{A}_4)$,

$2+\underline{\text{maxset}}(\bar{A}_3 \cap \bar{A}_4)$,

$3+\underline{\text{maxset}}(\bar{A}_1 \cap \bar{A}_3 \cap \bar{A}_4)$,

$3+\underline{\text{maxset}}(\bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4)\}$.

4.1.2.5: $(w_i, w_j) \notin E$ for $i \neq j$.

Let maxset = $\max\{1 +$ maxset$(V - \{v\} - A(v))$,

$2 +$ maxset$(\bar{A}_1 \cap \bar{A}_2)$,

$2 +$ maxset$(\bar{A}_1 \cap \bar{A}_3)$,

$2 +$ maxset$(\bar{A}_1 \cap \bar{A}_4)$,

$2 +$ maxset$(\bar{A}_2 \cap \bar{A}_3)$,

$2 +$ maxset$(\bar{A}_2 \cap \bar{A}_4)$,

$2 +$ maxset$(\bar{A}_3 \cap \bar{A}_4)$,

$3 +$ maxset$(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3)$,

$3 +$ maxset$(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_4)$,

$3 +$ maxset$(\bar{A}_1 \cap \bar{A}_3 \cap \bar{A}_4)$,

$3 +$ maxset$(\bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4)$,

$4 +$ maxset$(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4)\}$.

4.2: $d(w) \geq 5$ for some vertex w.

Let v,w be such that $d(v) = 4$, $d(w) \geq 5$, $(v,w) \in E$.

Let maxset = $\max\{1 +$ maxset$(V - \{w\} - A(w))$,

maxset$(V - \{w\})$.

Note that $V - \{w\}$ contains a vertex of degree three and all vertices are of degree three or greater.

5: $d(w) = 5$ for all vertices w.

5.1: $|V| = 6$.

Let maxset = 1.

5.2: $|V| > 6$.

Let maxset = max{1 + maxset(V - {v} -A(v)),

maxset(V - {v}) .

Note that V- {v} contains a vertex of degree four, a vertex of degree five, and all vertices are of degree four or greater.

6: Some vertex w has d(w) ≥ 6.

Let maxset = max{1 + maxset(V - {w} - A(w)) ,

maxset(V- {w})} .

end maxset.

3. Resource Bounds.

Let T(n) be an upper bound on the worst-case running time of maxset(V) when $|V| = n$. Let $T_i(n)$ be an upper bound on the worst-case running time of maxset(V) *when $|V| = n$ and case i occurs at the outermost level of recursion. Let p(n) be a polynomial which bounds the running time of the outermost level of recursion, exclusive of recursive calls. We have the following inequalities. (Starred inequalities are the worst cases.)

$$T_0(n) \le \max\left\{ \sum_{i=1}^{k} T(n_i) \mid \sum_{i=1}^{k} n_i = n \ , \ 1 \le n_i \le n \right\} + p(n) \ .$$

$$T_1(n) \le T(n-2) + p(n) \ .$$

$$T_{2.1}(n) \le p(n) \ .$$

$$T_{2.2}(n) \le T(n-3) + p(n) \ .$$

$$T_{2.3}(n) \le T(n-3) + T(n-5) + p(n) \ . \ ^*$$

$$T_{3.1}(n) \le T(n-4) + p(n) \ .$$

$$T_{3.2}(n) \le T(n-4) + T(n-5) + p(n) \ .$$

$$T_{3.3.1}(n) \le T(n-4) + T(n-6) + p(n) \ .$$

$$T_{3.3.2}(n) \le T(n-4) + 2T(n-7) + p(n) \ . \ ^*$$

$$T_{3.4.1}(n) \le T(n-4) + T(n-7) + p(n) \ .$$

$$T_{3.4.2.1}(n) \le T(n-4) + T(n-8) + p(n) \ .$$

$$T_{3.4.2.2}(n) \le T(n-4) + T(n-6) + T(n-8) + p(n) \ . \ ^*$$

$$T_{3.4.3.1}(n) \leq T(n-4) + T(n-10) + p(n) .$$

$$T_{3.4.3.2}(n) \leq T(n-4) + T(n-8) + T(n-10) + p(n) .$$

$$T_{3.4.3.3}(n) \leq T(n-4) + 2T(n-8) + T(n-10) + p(n) .$$

$$T_{3.4.3.4.1}(n) \leq T(n-4) + 2T(n-8) + T(n-10) + p(n) .^{*}$$

$$T_{3.4.3.4.2}(n) \leq T(n-4) + 4T(n-10) .^{*}$$

$$T_{3.4.3.4.3}(n) \leq T(n-4) + T(n-8) + 3T(n-11) .^{*}$$

$$T_{3.4.3.4.4}(n) \leq T(n-4) + 2T(n-9) + 2T(n-12) .^{*}$$

$$T_{3.4.3.4.5}(n) \leq T(n-4) + 3T(n-10) + T(n-13) .$$

$$T_{4.1.1.1}(n) \leq T(n-2) + T(n-6) + p(n) .^{*}$$

$$T_{4.1.1.2.1}(n) \leq T(n-2) + T(n-8) + p(n) .$$

$$T_{4.1.1.2.2}(n) \leq T(n-2) + 2T(n-8) + p(n) .^{*}$$

$$T_{4.1.1.2.3}(n) \leq T(n-2) + 2T(n-8) + p(n) .$$

$$T_{4.1.1.2.4}(n) \leq T(n-2) + T(n-8) + 2T(n-10) + p(n) .^{*}$$

$$T_{4.1.2.1}(n) \leq p(n) .$$

$$T_{4.1.2.2}(n) \leq T(n-5) + 3T(n-9) + p(n) .$$

$$T_{4.1.2.3}(n) \leq T(n-5) + 4T(n-9) + p(n) .^{*}$$

$$T_{4.1.2.4}(n) \leq T(n-5) + 4T(n-10) + T(n-11) + 2T(n-13) + p(n) .$$

$$T_{4.1.2.5}(n) \leq T(n-5) + 6T(n-11) + 4T(n-14) + T(n-17) + p(n) .^{*}$$

$$T_{4.2}(n) \leq T_3(n-1) + T(n-6) + p(n)$$

$$\leq \max\{T(n-5) + T(n-6) , T(n-5) + 2T(n-8) , T(n-5) + T(n-7) + T(n-9) ,$$

$$T(n-5) + 2T(n-9) + T(n-11) , T(n-5) + 4T(n-11) ,$$

$$T(n-5) + T(n-9) + 3T(n-12) , T(n-5) + 2T(n-10) + 2T(n-13)\}$$

$$+ T(n-6) + p(n) .$$

$$T_{5.1}(n) \leq p(n) .$$

$$T_{5.2}(n) \leq T_{4.2}(n-1) + T(n-6) + p(n)$$

$$\leq \max\{T(n-6) + T(n-7) , T(n-6) + 2T(n-9) , T(n-6) + T(n-8) + T(n-10) ,$$

$$T(n-6) + 2T(n-10) + T(n-12) , T(n-6) + 4T(n-12) ,$$

$$T(n-6) + T(n-10) + 3T(n-13) , T(n-6) + 2T(n-11) + 2T(n-14)\}$$

$$+ T(n-6) + p(n)$$

$$T_6(n) \leq T(n-1) + T(n-7) + p(n) .$$

$$T(n) \leq \max_i T_i(n) .$$

From each of the recursive bounds

$$T_i(n) \leq \sum_{i=1}^{k} a_i T(n-b_i) + p(n)$$

we get a polynomial equation

$$x^{b_k} = \sum_{i=1}^{k} a_i x^{b_k - b_i} .$$

If y is the maximum of the positive solutions to all these equations, $cy^{n+\epsilon}$ is a bound on the running time of the algorithm. It happens that the value of y is slightly less than $3/2$. By means of a tedious calculation using Table 1, one can prove by induction that $T(n) \leq c2^{n/3}$

without solving lots of polynomials. The constant c depends upon $p(n)$. The worst cases of the recursion are 4.1.1.2.4 and 4.1.2.5.

The storage required by the algorithm is certainly polynomial, since the depth of recursion is only $O(n)$. With careful programming, the storage required can be made linear in the size of the graph.

n	$2^{n/3}$
1	1.2599+
2	1.5876+
3	2.0000
4	2.5198+
5	3.1747+
6	4.0000
7	5.0397+
8	6.3496+
9	8.0000
10	10.079+
11	12.699+
12	16. 000
13	20.158+
14	25.398+
15	32.000
16	40.317+

Table 1. Fractional Exponentials for Inductive Proof of Time Bound.

4. Conclusions

We have presented a recursive algorithm which finds a maximum independent set in a graph of n vertices in $O(2^{n/3})$ time. The algorithm is an extension and improvement of one described in [10]. Though the case analysis used is lengthy, the algorithm could be programmed easily, and we believe the algorithm would perform well in practice.

Nevertheless, the main interest of the result is theoretical; it shows that even for NP-complete problems it is sometimes possible to develop algorithms which are substantially better in the worst case than the obvious enumeration algorithms. Whether the algorithm presented here can be improved substantially, and whether similar algorithms can be developed for other NP-complete problems, are open questions.

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