ON COMPLETE SUBGRAPHS OF r-CHROMATIC GRAPHS

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Abstract

Denote by $G(p,q)$ a graph of p vertices and q edges. $K_r = G(r, \binom{r}{2})$ is the complete graph with r vertices and K,(t) is the complete r chromatic (i.e., r-partite) graph with t vertices in each color class. G,(n) denotes an r-chromatic graph, and 6(G) is the minimal degree of a vertex of graph G . Furthermore denote by f,(n) the smallest integer so that every G,(n) with $\delta(G_r(n)) > f_r(n)$ contains a K_r . It is easy to see that $\lim_{n\to\infty} f_r(n)/n = c_r$ exists. We show that $c_4 > 2 + \frac{1}{9}$ and $c_r \geq r-2+\frac{1}{2} - \frac{1}{2(r-2)}$ for $r > 4$. We prove that if $\delta(G_3(n)) \geq n+t$ then G contains at least t^3 triangles but does not have to contain more than $4t^3$ of them.

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On Complete Subgraphs of r-chromatic Graphs

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1. Introduction

Denote by $G(p,q)$ a graph of p vertices and q edges. $K_r = G(r, \binom{r}{2})$ is the complete graph with r vertices and $K_r(t)$ is the complete r-chromatic (i.e., r-partite) graph with t vertices in each color class. $f(n\,;\,G(p,q))$ is the smallest integer for which every $G(n\,;\,f(n\,;\,G(p,q)))$ contains a $G(p,q)$ as a subgraph. In 1940, Turán [9] determined $f(n\,;\,K_r)$ for every $r > 3$ and thus started the theory of extremal problems on graphs. Recently many papers have been published in this area ([1],[2],[3],[4],[5],[6]).

In this paper we investigate r-chromatic graphs. We obtain some results that seem interesting to us and get many unsolved problems that we hope are both difficult and interesting.

$G_r(n)$ denotes an r-chromatic graph with color classes C_i, $|C_i| = n$, $i = 1,\ldots,r$. Here and in the sequel $|X|$ denotes the number of elements in a set X. A q-set or q-tuple is a set with q elements. $e(G)$ is the number of edges of a graph G and $\delta(G)$ is the minimal degree of a vertex of G. As usual, $[x]$ is the largest integer not greater than x.

At the Oxford meeting on graph theory in 1972, P. Erdös [7] conjectured that if $\delta(G_r(n)) > (r-2)n+1$ then $G_r(n)$ contains a K_r. Graver found a simple and ingenious proof for $r = 3$ but for $r \geq 4$ counterexamples were found. This discouraged further investigations but we hope to convince the reader that interesting and fruitful problems remain.

We prove that if $\delta(G_3(n)) \geq n+t$ then G contains at least t^3 triangles but does not have to contain more than $4t^3$ of them. For $n \geq 5t$ probably $4t^3$ is exact-but we prove this only for $t = 1$.

It is probably true that if $\delta(G_3(n)) > n + C\, n^{1/2}$ (C is a sufficiently large constant) then G contains a $K_3(2)$. We can prove only that $\delta(G_3(n)) > n+C\, n^{3/4}$ ensures the existence of a $K_3(2)$ subgraph of $G_3(n)$. More generally we obtain fairly accurate results on the magnitude of the largest $K_3(s)$ which every $G_3(n)$ with $\delta(G_3(n)) \geq n+t$ must contain, but many unsolved problems of a technical nature remain.

Our results on $G_r(n)$'s for $r > 3$ are much more fragmentary. Denote by $f_r(n)$ the smallest integer so that every $G_r(n)$ with $\delta(G_r(n)) > f_r(n)$ contains a K_r. It is easy to see that $\lim_{n\to\infty} f_r(n)/n = c_r$ exists. We show that $c_4 \geq 2+\frac{1}{9}$ and

$c_r \geq r-2 + \frac{1}{2} - \frac{1}{2(r-2)}$ for $r > 4$. We conjecture $\lim_{r\to\infty}(c_r-r+2) = \frac{1}{2}$.

It is surprising that this problem is difficult; perhaps we overlooked a simple approach. We can not even disprove $\lim_{r\to\infty}(c_r-r+2) = 1$.

Analogously to the results of [6], though we can not determine c_r, we prove that every $G_r(n)$ with $\delta(G_r(n)) > (c_r+\varepsilon)n$ contains at least $\eta\, n^r$ K_r's. We do not obtain interesting results for $\delta(G_r(n)) \geq n+t$, $t = a(n)$ for $r \geq 4$ though we believe they exist. As a slight extension of Turán's theorem, we determine the minimal number of edges of a $G_r(n)$ that ensures the existence of a K_ℓ, $3 \leq \ell \leq r$.

2. 3-chromatic Graphs.

Recall that $G_3(n)$ is a 3-chromatic graph with color classes C_i, $|C_i| = n$, $i \in Z_3$. For $x \in C_i$ let $D^+(x)$ (resp. $D^-(x)$) be the set of vertices of C_{i+1} (resp. C_{i-1}) that are joined to x. Put $d^+(x) = |D^+(x)|$, $d^-(x) = |D^-(x)|$. $d(x) = d^+(x) + d^-(x)$ is the degree of x in $G_3(n)$.

We shall frequently make use of the following trivial observation that we state as a lemma.

Lemma 1. Suppose $x \in C_i$, $y \in C_{i-1}$, and xy is an edge. Then there are at least

$$d^+(x) + d^-(y) - n$$

triangles containing the edge xy. There are at least

$$\sum_{y \in D'} (d^+(x) + d^-(y) - n)$$

triangles with vertex x, where $D' \subset D^-$.

Theorem 1. Let $G = G_3(n)$ have minimal degree at least $n+1$. Then G contains at least $\min(4,n)$ triangles and this result is best possible.

Proof. Put $d_i^+ = \max\{d^+(x) : x \in C_i\}$, $d_i^- = \max\{d^-(x) : x \in C_i\}$. We can suppose without loss of generality that $d_1^+ > d_2^+$ and $d_1^+ \geq d_3^+$. Let $x_1 \in C_1$, $d^+(x_1) = d_1^+$. Note that $d^+(x) + d^-(x) \geq n+1$ for every vertex x.

Suppose $d_1^+ \leq n-1$ and let $z \in D^-(x_1)$. If $d^+(z) = n-1$ then by Lemma 1 there are at least 2 triangles with vertex z. If $d^+(z) < n-1$ then again by Lemma 1 at least 2 triangles of G contain the edge $x_1 z$. Thus at least 2 triangles contain each vertex of $D^-(x_1)$ so G has at least $2|D^-(x_1)| \geq 4$ triangles.

Suppose now that $d_1^+ = n$ and the theorem holds for smaller values of n. Let us assume that G does not contain triangles T_1, T_2 such that $d^+(x_i) = n$ for a vertex of T_i, $i = 1,2$. Then Lemma 1 implies that $D^-(x_1)$ consists of a single vertex, say $D^-(x_1) = \{z_1\}$, and $d^+(z_1) = n$, $d^-(z_1) = 1$. Let $D^-(z_1) = \{y_1\}$. Then similarly $d^+(y_1) = n$ and $D^-(y_1) = \{x_1\}$, otherwise we have 2 triangles with the forbidden properties. Let $G' = G_3(n-1) = G - \{x_1, y_1, z_1\}$. In G' every vertex has degree at least n, so G' contains at least $n-1$ triangles and G contains at least n triangles. Thus, in proving the theorem, we can suppose without loss of generality that G contains triangles T_1, T_2 such that $d^+(x_i) = n$ for a vertex x_i of T_i, $i = 1,2$. Analogously, we can assume that G contains triangles T_1', T_2' such that $d^-(x_i') = n$ for a vertex x_i' of T_i', $i = 1,2$.

Let us show now that either these 4 triangles are all distinct or G contains at least n triangles. This will complete the proof of the assertion that G has at least $\min(4,n)$ triangles.

Let $x_1x_2x_3$ be a triangle of G, $x_i \in C_i$, $d^+(x_1) = n$. If $d^-(x_1) = n$ then for every edge yz, $y \in C_2$, $z \in C_3$, xyz is a triangle. As there are at least n such edges, G contains n triangles. If $d^-(x_2) = n$ then G contains at least n triangles with vertex x_3. Finally if $d^-(x_3) = n$, G has n triangles containing the edge x_1x_3. This completes the proof of the fact that G has at least $\min(4,n)$ triangles.

Let us prove now that the results are best possible. For $n = 1$ the triangle is the only graph satisfying the conditions. Suppose $G_{n-1} = G_3'(n-1)$ has minimal degree at least n (≥ 2) and contains exactly n-1 triangles. Let the color classes of G_{n-1} be C_i', $i \in Z_3$. Construct a graph $G_n = G_3(n)$ as follows. Put $C_i = C_i' \cup \{x_i\}$ and join x_i to every vertex of C_{i+1}. Then G_n has the required properties and contains exactly n triangles.

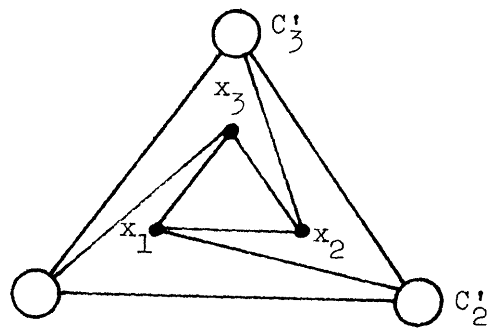


Figure 1

To complete the proof of Theorem 1 we show that for every $t \geq 1$ and $n \geq 5t$ there exists a tripartite graph $H(n,t) = G_3(n)$ with minimal degree $n+t$ that contains exactly $4t^3$ triangles. (For the proof of Theorem 1 the existence of the graphs $H(n,1)$, $n > 5$, is needed.)

We construct a graph $H(n,t)$ as follows. Let the color classes be C_i, $|C_i| = n$, $i \in Z_3$.

Let $A_i \subset C_i$, $|A_i| = n-2k$, $B_i = C_i - A_i$, $i \in Z_3$, and $B_1 = \bar{B}_2 \cup \bar{B}_3$, $|\bar{B}_j| = k$, $j = 2,3$.

Join every vertex of A_1 to every vertex of $A_2 \cup A_3$, join every vertex of $\bar{B}_j$ to every vertex of C_j, $j = 2,3$, and join every vertex of B_i to every vertex of C_j for $i = 2$, $j = 3$ and $i = 3$, $j = 2$. Finally, join every vertex of $\bar{B}_i$ to k arbitrary vertices of A_j for $i = 2$, $j = 3$ and $i = 3$, $j = 2$. (In Figure 2, a continuous line denotes that all the vertices of the corresponding classes are joined, and a dotted line means that every vertex of $\bar{B}_i$ is joined to k vertices of the other class.)

It is easily checked that the only triangles contained in H(n,k) are of the form $x_i y_i z_j$, $x_i \in \bar{B}_i$, $y_i \in B_i$, $z_j \in A_j$, $i = 2$, $j = 3$ and $i = 3$, $j = 2$. This shows that H(n,k) contains exactly $4k^3$ triangles. The proof of Theorem 1 is complete.

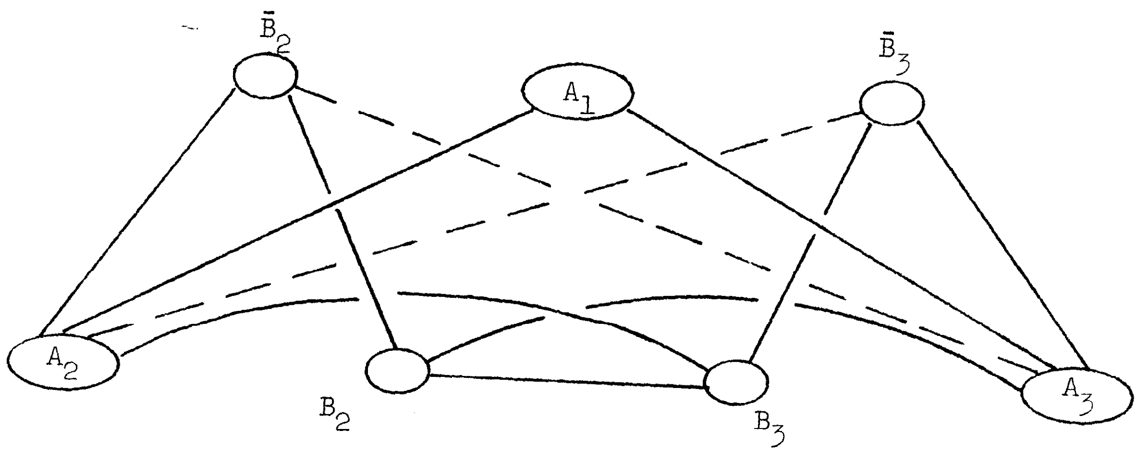


Figure 2

It is very likely that every graph $G_3(n)$, $n \geq 5t$, with minimal degree $n+t$ contains at least $4t^3$ triangles, i.e., that the graphs $H(n,t)$ have the minimal number of triangles with a given minimal degree. Though we can not show this, we can prove that t^3 is the proper order of the minimal number of triangles.

Theorem 2. Suppose every vertex of $G = G_3(n)$ has degree at least $n+t$, $t \le n$. Then there are at least t^3 triangles in G.

Proof. We can suppose without loss of generality that for some subset T_1 of C_1, $|T_1| = t$, we have

$$s = \sum_{x \in T_1} d^-(x) \ge \sum_{y \in T} d^+(y)$$

for all $T \subset C_i$, $|T| = t$, $i \in Z_3$.

Note that $d^-(x) \ge n+t-d^-(x)$ for every vertex x. For $x \in C_1$ let $T_x \subset D^-(x)$, $|T_x| = t$. Then by Lemma 1 the number of triangles of G containing one vertex of T_1 is at least

$$\sum_{x \in T_1} \sum_{y \in T_x} (d^+(x) + d^-(y) - n)$$

$$\ge \sum_{x \in T_1} \sum_{y \in T_x} (t + d^+(x) - d^+(y)) \ge \sum_{x \in T_1} (t^2 + td^+(x) - \sum_{y \in T_x} d^+(y))$$

$$\ge \sum_{x \in T_1} (t^2 + td^+(x) - s) \ge t^3 + ts - ts = t^3 .$$

Theorem 2 will be used to show the existence of large subgraphs $K_3(s)$ in a $G_3(n)$, provided $\delta(G_3(n)) \ge n+t$. First we need a simple lemma.

Lemma 2. Let $X = \{1, \ldots, N\}$, $A_i \subset X$, $i \in Y = \{1, \ldots, p\}$, $\sum_1^p |A_i| = pwN$ and $(1-\alpha)wp \ge q$, $0 < \alpha < 1$, where N, p, q and r are natural numbers. Then there are q subsets $A_{i_1}, \ldots, A_{i_q}$ such that

$$\left| \bigcap_{t=1}^{q} A_{i_t} \right| \ge N(\alpha w)^q .$$

Proof. For $i \in X$ let $Y_i = \{j: i \in A_j, j \in Y\}$, $y_i = |Y_i|$. We say that a q-set τ of Y belongs to $i \in X$ if $i \in \bigcap_{j \in \tau} A_j$. Clearly $\binom{y_i}{q}$ q-sets belong to $i \in X$. As $\sum_1^N y_i \geq pwN$,

$$\sum_1^N \binom{y_i}{q} \geq N\binom{wp}{q} \geq N\binom{p}{q}\binom{wp}{q}\Big/\binom{p}{q} \geq \binom{p}{q}N(\alpha w)^q .$$

Thus at least one q-set of Y belongs to at least $N(\alpha w)^q$ elements of X and this is exactly the assertion of the lemma.

The following immediate corollary is essentially a theorem of Kövári, Sós and Turán [8].

Corollary 1. Let $n^{1-1/s} \geq s$. Then every graph G with n vertices and at least $n^{2-1/s}$ edges contains a $K_2(s)$.

Proof. Let X be the set of vertices of G, let A_i be the set of vertices joined to the i-th vertex. Put $w = 2n^{-1/s}$, $\alpha = 1/2$, $q = s$, and apply the lemma.

Theorem 3. Suppose $\delta(G_3(n)) \geq n+t$, and s is an integer and
$$s \leq \left[\left(\frac{\log 2n}{\log n - \log t + (\log 2)/3}\right)^{1/2}\right].$$ Then $G_3(n)$ contains a $K_3(s)$.

Proof. Let $Y = C_1 = \{1,\dots,n\}$ and let X be the set of n^2 pairs (x,y), $x \in C_2$, $y \in C_3$. Let A_i be the set of pairs $(x,y) \in X$ for which (i,x,y) is a triangle of $G_3(n)$. As by Theorem 2 the graph contains at least t^3 triangles, Lemma 2 implies that there exist s vertices of C_1, say $1,2,\dots,s$, such that

$$|E| = |\bigcap_1^s A_i| \geq n^2(t^3/(2n^3))^s \geq (2n)^{2-1/s}$$

Thus, by Corollary 1, the graph with vertex set $C_2 \cup C_3$ and edge set E contains a $K_2(s)$. This $K_2(s)$ and the vertices $1,2,\ldots,s$ of C_1 form a $K_3(s)$ of $G_3(n)$, as claimed.

Corollary 2. Let $n > 2^8$ and suppose $\delta(G_3(n)) \geq n + 2^{-1/2} n^{3/4}$. Then $G_3(n)$ contains a $K_3(2)$.

As we remarked in the introduction, it seems likely that already $\delta(G_3(n)) > n + cn^{1/2}$ ensures that $G_3(n)$ contains a $K_3(2)$.

Theorem 4. Suppose $\delta(G_3(n)) \geq n+t$. Let $S = \left[\frac{\log 2n}{3(\log 2n - \log t)}\right]$ and

$$s \leq \min\left\{\frac{t^3}{4n^2} 2^{-2S}, \frac{t^3}{4n^3} S\right\}$$

Then $G_3(n)$ contains a $K_3(s)$.

Proof. The graph $G_3(n)$ contains at least t^3 triangles. Thus there are at least $\frac{t^3}{2n}$ edges xy, $x \in C_2$, $y \in C_3$, such that each of them is on at least $\frac{t^3}{2n^2}$ triangles. Let H be the subgraph spanned by the set E of the edges. Then, by Corollary 1, H contains a $K = K_2(S)$, say with color classes $C_2^* \subset C_2$ and $C_3^* \subset C_3$, since

$$(2n)^{2-1/S} \leq \frac{t^3}{2n}.$$

Let us say that a vertex $x \in C_1$ and an edge e of K correspond to each other if a triangle of $G_3(n)$ contains both of them. As by the construction at least $\frac{t^3}{2n^2}$ vertices correspond to an edge of K, there is a set $C_1^* \subset C_1$, $|C_1^*| \geq \frac{t^3}{4n^3} S^2$ edges of K.

Look at a vertex $x \in C_1^*$ and at the endvertices of the edges to which it corresponds. The set of endvertices can be chosen in at most 2^{2s} ways so there is a set $B_1 \subset C_1^*$ of at least

$$\frac{t^3}{4n^2} 2^{-2s} > s$$

vertices which correspond to the same endvertex set $B_2 \cup B_3$, $B_2 \subset C_2^*$, $B_3 \subset C_3^*$. Clearly

$$\min(|B_2|, |B_3|) \geq \frac{t^3}{4n^3} s^2/s = \frac{t^3 s}{4n^3} \geq s ,$$

and $G_3(n)$ --contains the complete tripartite graph with vertex classes B_1, B_2, B_2.

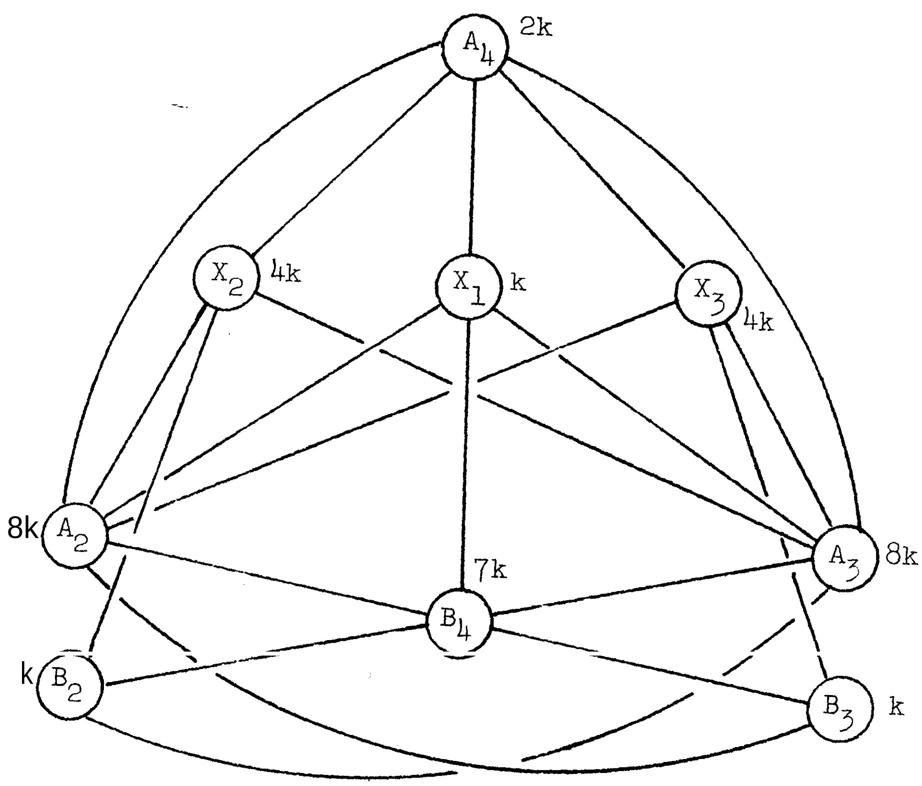
Corollary 3. Let $\delta(G_3(n)) \geq n + c \frac{n}{(\log n)^a}$, where $c > 0$ and $\alpha > 0$ are constants. Then there is a constant $C = C(c,\alpha)$ for which $G_3(n)$ contains a $K_3(s)$ with $s \geq C(\log n)^{1-3\alpha}/\log \log n$.

3. r-chromatic Graphs.

Let now $G_r(n)$ be an r-chromatic graph with color classes C_i, $|C_i| = n$, $i = 1,\dots,r$. One could hope (see [7]) that if every vertex of a $G_r(n)$ is of degree at least $(r-2)n+1$ then the graph contains a K_r. However, this is not true for $r \geq 4$ and sufficiently large values of n.

Let $n = qk$, $k \geq 1$, and construct a graph $F_4(n) = G_4(n)$ as follows. Let $C_1 = X_1 \cup X_2 \cup X_3$, $|X_1| = k$, $|X_2| = |X_3| = 4k$, $C_i = A_i \cup B_i$, $|A_i| = 8k$, $|B_i| = k$, $i = 2,3$, and $C_4 = A_4 \cup B_4$,

$|A_4| = 2k$, $|B_4| = 7k$. Join every vertex of X_1 to every vertex of $A_2 \cup A_3 \cup C_4$; join every vertex of X_i to every vertex of $C_i \cup A_j \cup A_4$, $i,j = 2,3$, $i \neq j$; join every vertex of A_4 to every vertex of $A_2 \cup A_3$; join every vertex of B_4 to every vertex of $C_2 \cup C_3$; and, finally join every vertex of A_i to every vertex of B_j , $i,j = 2,3$, $i \neq j$. The obtained graph is $F_4(n)$ (see Figure 3).



$F_4(n)$

Figure 3

Clearly every vertex of $F_4(n)$ has degree at least $19k = \left(2 + \frac{1}{9}\right) n$. Furthermore, the triangles in $F_4(n) - C_4$ are of the form xyz , where $x \in X_2$, $y \in B_2$, $z \in A_3$ or $x \in X_3$, $y \in A_3$, $z \in B_2$. As no vertex of C_4 is joined to all 3 vertices of such a triangle, $F_4(n)$ does not contain a K_4 . This example shows that if the minimal degree in a $G_4(n)$ is at least $\left(2 + \frac{1}{9}\right) n$ then $G_4(n)$ does not necessarily contain a K_4 .

Let now $r \geq 5$, $k \geq 1$ and $n = 2(r-2)k$. Construct a graph $F_r(n) = G_r(n)$ as follows. Let $C_i = A_i \cup B_i$, $|A_i| = |B_i| = (r-2)k = n/2$, let $C_{r-1} = \bigcup_1^{r-2} A^j$, $|A^j| = 2k$, $C_r = \bigcup_1^{r-2} B^j$, $|B^j| = 2k$, $i,j = 1,\ldots,r-2$. Join two vertices of $\bigcup_1^r C_i$ that are in different classes unless one vertex is in A_i and the other in $B_{i+1} \cup A^i$, or one vertex is in B_i and the other in $A_{i+1} \cup B^i$, $i = 1,\ldots,r$, where $A_{r+1} \equiv A_1$, $B_{r+1} \equiv B_1$. In the obtained graph $F_r(n)$ clearly every vertex has degree at least $\left(\frac{1}{2} - \frac{1}{r-2}\right)$. Furthermore, if $K = K_{r-2} \subset F_r(n) - C_{r-1} \cup C_r$ then either each A_i $(i = 1,\ldots,r-2)$ or each B_i $(i = 1,\ldots,r-2)$ contains a vertex of K . As no vertex of C_{r-1} is joined to a vertex in each A_i $(i = 1,\ldots,r-2)$ and no vertex of C_r is joined to a vertex in each B_i $(i = 1,\ldots,r-2)$, the graph $F_r(n)$ does not contain a K_r .

Denote by $t_k(n)$ the maximum number of edges of a k-chromatic graph. Turán's theorem [9] states that $f(n,K_p) = t_{p-1}(n) + 1$. This result has the following immediate extension to r-chromatic graphs.

Theorem 5. $\max\{e(G_r(n)): G_r(n) \not\supset K_p\} = t_{p-1}(r)n^2$.

Proof. Suppose $G = G_r(n)$ does not contain a K_p . Let H be a subgraph of G spanned by r vertices of different classes. Then H contains at most $t_{p-1}(r)$ edges. Furthermore, there are n^r such subgraphs H and every edge of G is contained in n^{r-2} of them. Thus G has at most $t_{p-1}(r)n^2$ edges.

The graph $G_r(n)$ obtained from a maximal (p-1) -chromatic graph by replacing each vertex by a set of n vertices has exactly $t_{p-1}(r)n^2$ edges and does not contain a K_p .

Corollary 4. Suppose $\delta(G_r(n)) \geq \delta$. If $t_{p-1}(r)n < \frac{r\delta}{2}$ then $G_r(n)$ contains a K_p . In particular, $f_r(n) \leq (r-2 + \frac{r-2}{r})n$ so $c_r = \lim_{n\to\infty} f_r(n)/n < r-2 + \frac{r-2}{r}$.

Theorem 6. Let $\varepsilon > 0$ and $\delta(G_r(n)) \geq (c_r+\varepsilon)n$. Then there is a constant $\delta_\varepsilon > 0$, depending only on ε , such that $G_r(n)$ contains at least $\delta_\varepsilon n^r$ K_r's.

Proof. Let $m > m_0(\varepsilon)$ be an integer. We shall prove that for all but $\eta\binom{n}{m}^r$ ($\eta > 0$ is independent of m) choices of m-tuples from the sets C_i the subgraph $G_r(m)$ of $G_r(n)$ spanned by the r m-tuples contains a K_r . (The total number of choices of the m-tuples is $\binom{n}{m}^r$.) This assertion naturally implies that our graph contains at least

$$(1-\eta)\binom{n}{m}^r / \binom{n-1}{m-1}^r = (1+o(1))(1-\eta)n^r/m^r \qquad (*)$$

K_r's since at least $(1-\eta)(\binom{n}{m})^r$ K_r's are obtained and each of them occurs $\binom{n-1}{m-1}$ times. The relation (*) of course proves Theorem 6.

Let $x \in C_i$. Suppose x is joined to $c_j^{(x)}n$ vertices of C_j, $j \neq i$. As $cr > r-2$, $c_j^{(x)} > c > 0$ for absolute constant c. Call an m-tuple in C_j <u>bad with respect to</u> x if fewer than $(c_j^{(X)} - \frac{\varepsilon}{2r})m$ of the vertices of our m-tuple are joined to x. A simple and well known argument using inequalities of binomial coefficients gives that the number of bad m-tuples with respect to x is less than $(1-\eta)^m\binom{n}{m}$, where $\eta = \eta(\varepsilon,c) > 0$ is independent of m.

We call a vertex x and a bad m-tuple with respect to x a <u>bad pair</u>. Observe that if $\bigcup_1^r A_i$ $(A_i \subset C_i,\ |A_i| = m)$ does not contain a bad pair then the subgraph spanned by $\bigcup_1^r A_i$ contains a K_r since each of its vertices has degree greater than $(c_r + \varepsilon/2)m > f_r(m)$ if $m > m_o(\varepsilon)$. We now estimate by an averaging process the number of $\{A_i\}_1^r$ without a bad pair.

If (x,A_i), $x \in C_n$, is a bad pair there are clearly $\binom{n-1}{m-1}\binom{n}{m}^{r-2}$ sets $\{A_j\}_1^r$ which contain the bad pair. Thus if there are $\gamma\binom{n}{m}$ families $\{A_j\}_1^r$, $|A_j| = m$, $A_j \subset C_j$, $1 \leq j \leq r$, which contain a bad pair then the number of bad pairs is at least

$$\gamma\binom{n}{m}^r\binom{n-1}{m-1}\binom{n}{m}^{r-2} = \gamma \frac{n}{m}\binom{n}{m} .$$

On the other hand to a given vertex x there are fewer than $r(1-\eta)^m\binom{n}{m}$ bad sets thus the number of bad pairs is less than

$$nr^2(1-\eta)^m\binom{n}{m} .$$

Thus

$$\gamma < r^2 m(1-\eta)^m ,$$

which proves our theorem.

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