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A MACHINE-INDEPENDENT ALGOL PROCEDURE FOR ACCURATE FLOATING-POINT SUMMATION

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```
procedure sum (x, n, m, result, fail);
value n, m; integer n, m; real result;
array x; label fail;
begin comment This Algol 60 procedure is an implementation of the
```

floating-point summation technique described in Malcolm (1971). This implementation is machine-independent in the sense that it will work on any computer having a floating-point number system F characterized as follows: Each number $x \in F$ has a radix-β t-digit <u>fraction</u> where $t \geq 1$. The radix β can be any positive integer greater than 1. The exponent e is assumed to lie in the range

$$b \leq e \leq B ,$$

where $b \leq 0$ and $B > t$. Each nonzero $x \in F$ has the representation

$$x = \pm .d_1 d_2 \ldots d_t \cdot \beta^e ,$$

where $d_1, \ldots, d_t$ are integers satisfying

$$0 \leq d_i \leq \beta-1 , \quad (i=1, \ldots ,t) .$$

The number 0 is contained in F, but no assumption is made about its representation. All floating-point operations (e.g., addition and multiplication) are assumed to result in either 0 or a <u>normalized</u> floating-point number contained in F. The machine may do either proper rounding or chopping (truncation). (Note that this definition of F excludes machines using extra-length accumulators for intermediate arithmetic. However, this algorithm is seldom needed on such machines.)

The parameters β and t of F are automatically computed at execution time by a technique described in Malcolm (1972). Since the range of the floating-point exponent cannot be determined automatically, the input parameter m is used for allocating the set of accumulators used by the algorithm.

Provided no overflow or underflow occurs, and none of the x[i] are larger than 10^m, or smaller than 10^{-m}, in magnitude, and $n \leq \beta^{\ell+1}/16$, where $\ell = \lfloor t/2 \rfloor$, then

$$\text{result} \approx \sum_{i=1}^{n} x[i]$$

is returned with nearly full-precision accuracy. The bound on the relative error is given by Theorem 2 in Malcolm (1971) as

$$\lceil (t+1)/\lfloor \log_\beta 16 \rfloor \rceil \, \beta^{1-t} .$$

If any of the x[i] are larger than 10^m or smaller than 10^{-m}, then the error exit fail is taken. ;

```
Boolean rnd; integer beta, t, t2, nu, L, U;
procedure ENVRON (beta, t, rnd);
Boolean rnd; integer beta, t;
begin comment This procedure is an Algol 60 translation of the (first)
Fortran subroutine ENVRON given in Malcolm (1972). ;
      real a, b, e;

      for e := 2, 2×e while (a+1)-a=1 do a := e;
      for e := 2, 2×e while a+b=a do b := e;
      beta := (a+b)-a; rnd := a+(beta-1) > a; t := 0;
      for a := 1, beta×a while (a+1)-a=1 do t := t+1

end ENVRON;

ENVRON (beta, t, rnd); t2 := t÷2 ; nu := ln(16)/ln(beta);
U := entier (m×ln(10)/(ln(beta)×nu)) + 1;
L := entier((-m×ln(10)/ln(beta) - t2)/nu);
```

comment In the notation of Malcolm (1971), ℓ = t2 is the padding that each of the numbers added to the accumulators will have. Each of the x[i] will be split into two halves (i.e. q=2) having the last t2 digits equal zero. The variable nu above is used for ν defined in Equation (2) of Malcolm (1971). The value for nu computed above is rather arbitrary and was chosen to make nu sufficiently smaller than t2 . The variables U and L are the upper and lower bounds on the indices of the accumulators which are declared in the following block. They are chosen to allow the x[i] to range from 10^{-m} to 10^{m} in magnitude. In slightly different notation, they are

$$U = \lceil m/(\nu \times \ell og_{10}\beta) \rceil ,$$

$$L = \lfloor (-m/\ell og_{10}\beta - \lfloor t/2 \rfloor)/\nu \rfloor ;$$

```
begin array accumulators[L:U]; integer ex;
      real xL, xH;

      integer procedure e(x);
      value x; real x;
      begin comment This procedure computes the exponent e of the
          floating-point number x . ;
          real y, q; integer ex;
          x := abs(x); ex := 0; for y := 1,q
          while x>y do begin ex := ex+1; q := beta×y; end;
          for y := q, y/beta while x<y do ex := ex-1;
          e := ex
      end e;
```

```
comment  initialize the array of accumulators;
for  i:=L  step  1  until  U  do  accumulators[i] := 0;
comment  accumulate the nonzero x[i]s;
for  i:=1  step  1  until  n  do  if  x[i]≠0  then
begin   ex := e(x[i]);
    if  entier(ex/nu)>U ∨ ex-t2<L×nu  then  go to  fail;
    comment  Now the  x[i]  is split into a high- and low-order
    part, xH and xL.  The method used here is to add the proper
    power of  β  to  x[i]  to force it to preshift  t2  digits
    to the right and then either truncate or round the last  t2
    significant digits.  Then the same power of  β  is subtracted
    to cause a post normalization which brings in  t2  trailing
    zero digits.  The resulting high-order part of  x[i]  is then
    subtracted from  x[i]  to produce the low-order part such
    that the sum of the high- and low- order parts is exactly
    equal to  x[i].  This method of splitting a floating-point
    number into two halves is similar to that given by Dekker
    (1971).  ;
    xH := beta↑(ex-1+t2);  xH := (xH+x[i]) - xH;
    xL := x[i] - xH;
    comment  xH  and  xL  can now be added to the appropriate
    accumulators.  ;
    accumulators[entier(ex/nu)]  := xH;
    accumulators[entier((ex-t2)/nu)]  := xL
end;  comment  Now sum the accumulators in decreasing order.  ;
result := 0;
for  i:=U  step  -1  until  L  do
```

```
                result := result + accumulators[i]
        end
    end  sum
```

References

1. Dekker, T.J. (1971), "A Floating-Point Technique for Extending the Available Precision," Numer. Math. 18, 224-242.

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3. Malcolm, Michael A. (1972), "Algorithms to reveal properties of floating-point arithmetic," Comm. ACM, Vol. 15, No. 11, November, 949-951.