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ASYMPTOTIC BOUNDS FOR THE NUMBER OF CONVEX n-OMINOES

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Abstract

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Asymptotic Bounds for the Number of Convex n-Ominoes

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Introduction

Unit squares having their vertices at integer points in the Cartesian plane are called cells. A point set equal to a union of n distinct cells which is connected and has no finite cut set is called an n-omino. Two n-ominoes are considered the same if one is mapped onto the other by some translation of the plane. (Such n-ominoes were called fixed animals with n cells by R. C. Read [8].) For example, there are six different 3-ominoes as shown in Figure 1.

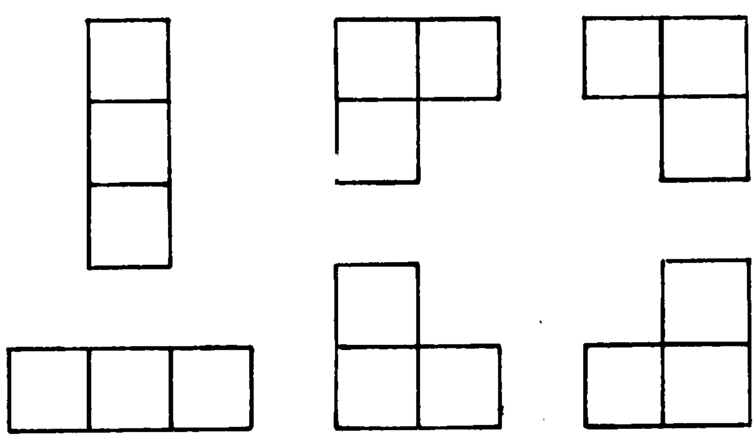

Figure 1. The 3-ominoes.

Let $t(n)$ denote the number of distinct n-ominoes. It is known [2] that the sequence $((t(n))^{1/n}: n = 1,2,\ldots)$ tends to a limit θ. The investigation of θ began with Eden's [1] work; he managed to prove that $3.14 < \theta \leq 6.75$. There has been considerable effort expended to improve these bounds. Currently, the best lower bound (given in [2]) is $3.72 < \theta$, while the best upper bound (given in [5]) is $\theta < 4.65$.

An n-omino is <u>row-convex</u> when each row of the n-omino is a connected strip of cells. <u>Column-convex</u> n-ominoes are defined analogously. All six of the 3-ominoes (shown in Figure 1) are both row-convex and column-convex; in general, such n-ominoes are said to be <u>row-column-convex</u>, or just convex for short. It was shown in [3] (and in [2] by a second method) that

$$(1) \qquad \frac{x(1-x)^3}{1-4x+7x^2-5x^3} = \sum_{n=1}^{\infty} b(n)x^n$$

where $b(n)$ denotes the number of distinct row-convex n-ominoes. (This result was also obtained by Polya [6].) Thus, it follows that the sequence $((b(n))^{1/n}: n = 1,2,\ldots)$ tends to a limit β which is equal to the largest real root of $y^3 - 4y^2 + 7y - 5 = 0$; that is, $\beta = 3.20 \ldots$.

Recently, Donald Knuth wrote us from his sabbatical hide-out in CENSORED (where he is secretly writing Volume 4 of his septuple, <u>The Art of Computer Programming</u>), and asked us if the number $c(n)$ of convex n-ominoes had been investigated. This paper is entirely motivated by Knuth's question. We shall be concerned with the problem of effectively calculating the limit γ of the sequence $((c(n))^{1/n}: n = 1,2,\ldots)$. One of the first things we prove is that this limit exists. Later on we show how to calculate upper and lower bounds for γ and give the best results obtained by these methods.

Existence of $\lim(n \to \infty)(c(n))^{1/n}$

Following Ceasar's admonition, we divide, then conquer. A convex n-omino may be split into three parts by making two cuts between certain rows so that the upper and lower parts are roughly trapezoids, and the middle part is roughly a parallelogram. A typical sectioning of this sort is shown in Figure 2. More precisely, the trisection of a convex n-omino A is accomplished by cutting along the lowest level of A where the left boundary of A goes to the right and by cutting along the lowest level of A where the right boundary of A goes to the left.

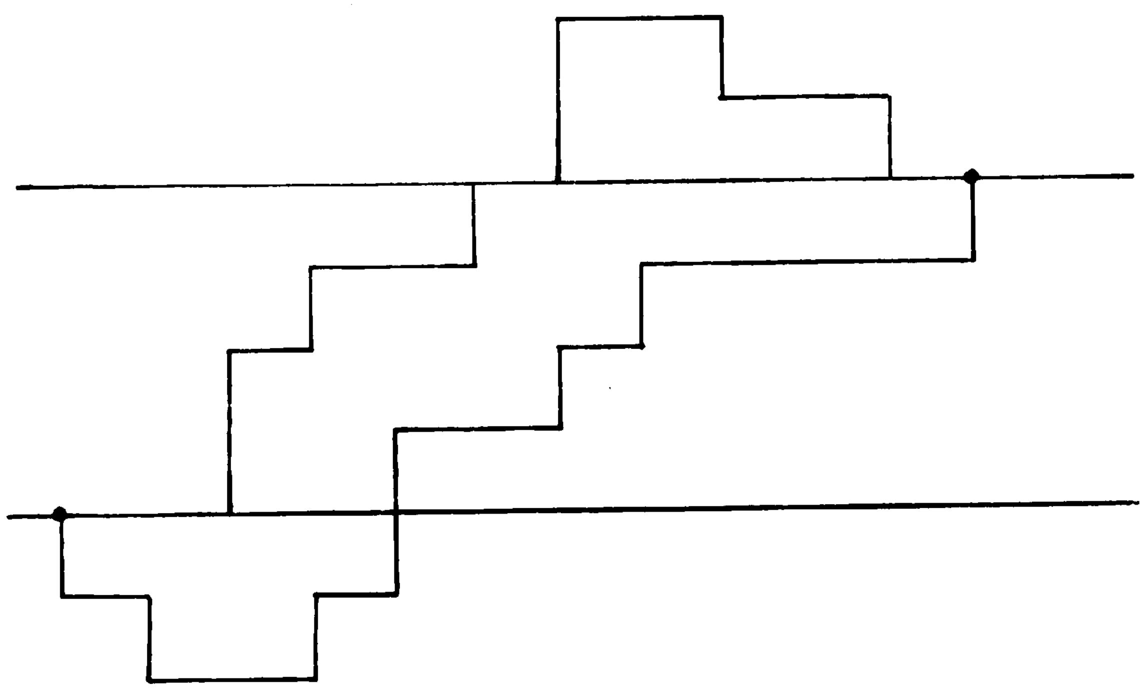

Figure 2. Trisection of a convex 28-omino.

A convex n-omino whose left boundary climbs to the right and whose right boundary climbs to the left corresponds to a partition of n called a stack by E. M. Wright [9]. We let $s(n)$ denote the number of distinct n-ominoes corresponding to stacks; for example, there are four 3-ominoes shown in Figure 1 which correspond to stacks, so $s(3) = 4$. A convex n-omino whose left and right boundaries both climb to the right is called a parallelogram, and $p(n)$ will denote the number of distinct n-ominoes which are parallelograms. Clearly, $p(n) \leq c(n)$ for all n ; also, $s(n) \leq p(n)$ for all n (the diagram in Figure 3 suggests a proof of this fact). Finally, an obvious construction establishes that $p(m)p(n) \leq p(m+n)$ for all m,n . Now we use the fact that if $\{u_n\}$ is a sequence of natural numbers such that $((u_n)^{1/n}: n = 1,2,\ldots)$ is bounded and $u_m u_n \leq u_{m+n}$ for all m,n , then $\lim(n \to \infty)(u_n)^{1/n}$ exists. (For similar results, see Pólya and Szego [7 , p. 171].) We have $p(n) \leq b(n) < (3.20)^n$ for all large n , and $p(m)p(n) \leq p(m+n)$, so

$$(2) \qquad \lim_{n \to \infty} (p(n))^{1/n} = \gamma$$

exists. Using the fact that every convex n-omino splits into two stacks and one parallelogram, we can reconstruct these n-ominoes by pasting together two stacks and one parallelogram in various ways. Again, using an obvious construction, and using the fact that $p(i)p(j)p(k) \leq p(i+j+k)$ for all i,j,k , it is easy to show that

$$(3)\qquad c(n) \le 2n^2 \sum_{(i,j,k)} s(i)p(j)s(k)$$

$$\le 2n^2 \sum_{(i,j,k)} p(i)p(j)p(k)$$

$$\le 2n^2\binom{n+2}{2}p(n) \le (n+2)^4 p(n)$$

where the index of summation in the sums extends over all compositions (i,j,k) of n into non-negative parts. There are $\binom{n+2}{2}$ such compositions.

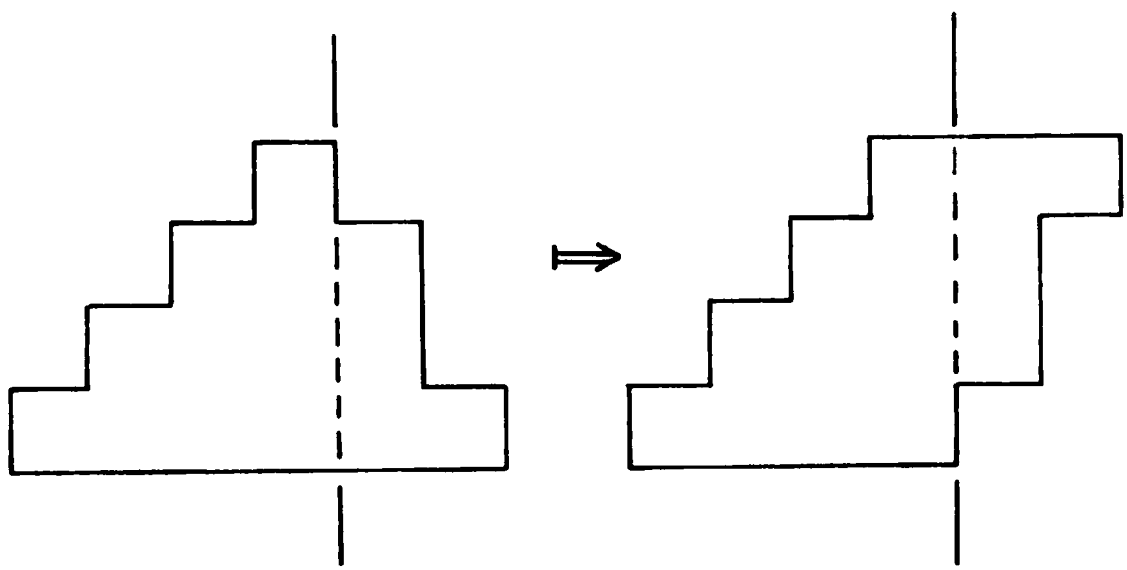

Figure 3. An injection showing $s(n) \le p(n)$.

Using (2) and (3) together with the fact that $p(n) \le c(n)$ for all n , we have

$$(4)\qquad \gamma = \lim_{n\to\infty} (p(n))^{1/n} \le \lim_{n\to\infty} \inf(c(n))^{1/n}$$

$$\le \lim_{n\to\infty} \sup(c(n))^{1/n} \le \lim_{n\to\infty} ((n+2)^4 p(n))^{1/n} = \gamma .$$

Hence, $\lim(n\to\infty)(c(n))^{1/n}$ exists, and

$$(5) \qquad \lim_{n\to\infty}(c(n))^{1/n} = \lim_{n\to\infty}(p(n))^{1/n} = \gamma \ .$$

An Integral Equation

We shall use a theory developed in [4] concerning a double sequence $(b(n,a): n,a = 1,2,\ldots)$ defined in terms of given sequences $(f(m,n): m,n = 1,2,\ldots)$ and $(g(n): n = 1,2,\ldots)$ as follows:

$$(6) \qquad b(n,a) = \sum f(a_1,a_2)f(a_2,a_3)\ldots f(a_{k-1},a_k)g(a_k)$$

where the index of summation extends over all k-tuples $(a_1,\ldots,a_k)$ of natural numbers for $k = 1,\ldots,n$ with $a_1 = a$ and $a_1 + \ldots + a_k = n$. It was shown that if

$$(7) \qquad G(x) = \sum_{n=1}^{\infty} g(n)x^n \quad ,$$

and

$$(8) \qquad F(x,y) = \sum_{m,n=1}^{\infty} f(m,n)x^m y^n \quad ,$$

converge for $|x|$ and $|y|$ sufficiently small, then

$$(9) \qquad B(x,y) = \sum_{n=1}^{\infty}\sum_{a=1}^{n} b(n,a)y^a x^n$$

converges for $|x|$ and $|y|$ sufficiently small, and

$$(10) \qquad B(x,y) = G(xy) + \frac{1}{2\pi i}\int_C F(xy, \frac{1}{s})\, B(x,s)\, \frac{ds}{s}$$

where C is a contour in the s-plane which includes $s = 0$ and the singularities of $F(xy, \frac{1}{s})$ but excludes the singularities of $B(x,s)$. The theory of (10) runs parallel to that of the Fredholm integral equation. In particular, if $F(x,y)$ has the special form

$$(11) \qquad F(x,y) = R_1(x)S_1(y) + \ldots + R_t(x)S_t(y) \quad ,$$

we say F is _separable_, and it turns out that (10) can be converted into a system of t equations linear in t unknown functions. The system can be solved and the solution yields a formula for $B(x,y)$. We shall give an example of this later on.

If F is not separable we can still get information about B by approximating F with something that is separable. Suppose

$$(12) \qquad K(x,y) = \sum k(m,n)x^m y^n$$

and $k(m,n) \leq f(m,n)$ for all m,n, then we say K is _a lower bound_ on F; an _upper bound_ on F is defined analogously. If K is separable, we may substitute K for F in (10) and calculate a lower bound for B. Upper bounds for B may be obtained in a similar fashion. We shall adopt this strategy too, so an example is forthcoming.

The relevance of the foregoing discussion to the enumeration of n-celled parallelograms is as follows: the number of (m+n)-celled parallelograms having m cells in one row and n cells in a second row is

$$(13) \qquad f(m,n) = \min\{m,n\} \quad .$$

It is fairly easy to show that the number of n-celled parallelograms with exactly k rows of cells having exactly a_i cells in the i-th row for $i = 1,\ldots,k$ is

$$(14) \qquad f(a_1,a_2)f(a_2,a_3)\ \cdots\ f(a_{k-1},a_k) \quad .$$

Thus, if we take f as defined in (13) and put $g(j) = 1$ for all j, we can sum (6) over $a = 1,\ldots,n$ and obtain $p(n)$. In this case, we have

$$(15) \qquad F(x,y) = \frac{xy}{(1-x)(1-y)(1-xy)} \quad ,$$

and

$$(16) \qquad G(x) = \frac{x}{1-x} \quad .$$

Substituting these functions in (10) gives

$$(17) \qquad B(x,y) = \frac{xy}{1-xy} + \frac{1}{2\pi i}\int_C \frac{xy\ B(x,s)ds}{(1-xy)(s-1)(s-xy)}$$

$$= \frac{xy}{1-xy} + \frac{xy}{(1-xy)^2}B(x,1) - \frac{xy}{(1-xy)^2}B(x,xy) \quad .$$

We can iterate (17) to eliminate $B(x,xy), B(x,x^2y),\ldots$ successively to find

$$(18) \qquad B(x,y) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^{k(k+1)/2}y^k(1 - x^ky + B(x,1))}{(1-xy)^2(1-x^2y)^2\cdots(1-x^ky)^2} \quad .$$

Setting $y = 1$ in (18), we solve for $B(x,1)$, the generating function of $(p(n): n = 1,2,\ldots)$, which turns out to be

$$(19) \qquad B(x,1) = \frac{\dfrac{x}{1-x} - \dfrac{x^3}{(1-x)^2(1-x^2)} + \dfrac{x^6}{(1-x)^2(1-x^2)^2(1-x^3)} - \cdots}{1 - \dfrac{x}{(1-x)^2} + \dfrac{x^3}{(1-x)^2(1-x^2)^2} - \dfrac{x^6}{(1-x)^2(1-x^2)^2(1-x^3)^2} + \cdots}$$

$$= \sum_{n=1}^{\infty} p(n)x^n \quad .$$

We have been unable to make use of (19) in estimating $p(n)$. Instead we use upper and lower bounds for F as defined in (15), and then use (10) to calculate upper and lower bounds for B .

Lower Bounds

Let

(20) $$F_k(x,y) = \sum_{m,n=1}^{k} f(m,n)x^m y^n$$

where $f(m,n) = \min\{m,n\}$ just as in (13), and let $B_k(x,y)$ denote the solution of (10) having F_k substituted for F . Since F_k is a lower bound for F , it follows that B_k is a lower bound for B . It was shown in [4] that when the kernel of (10) is approximated by a polynomial as in this case, then $B_k(x,1)$ is a rational function, say $B_k = P_k/Q_k$ with P_k and Q_k polynomials, and the denominator of B_k may be expressed as a determinant. In the present situation this turns out to be

(21) $$Q_k(x) = \begin{vmatrix} 1-x & 1 & 1 & \cdots & 1 \\ 1 & 2-x^2 & 2 & \cdots & 2 \\ 1 & 2 & 3-x^3 & \cdots & 3 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 2 & 3 & \cdots & k-x^k \end{vmatrix}$$

If we put $Q_0(x) = 1$ and $Q_1(x) = 1-x$ we can use (21) to verify that

(22) $$Q_k(x) = (1-x^{k-1}-x^k)Q_{k-1}(x) - x^{2k-2}Q_{k-2}(x)$$

for $k = 2,3,\ldots$. For example,

$$Q_2(x) = 1 - 2x - x^2 + x^3 \ ,$$

$$Q_3(x) = 1 - 2x - 2x^2 + 2x^3 + 2x^4 + x^5 - x^6 \ ,$$

$$Q_4(x) = 1 - 2x - 2x^2 + x^3 + 3x^4 + 5x^5 - 2x^6 - 2x^7 - 2x^8 - x^9 + x^{10} \ .$$

Letting γ_k denote the largest real root of $Q_k(1/x) = 0$, we have $\gamma_1 \le \gamma_2 \le \cdots \le \gamma$, where γ is defined in (2). We have used a computer to calculate lower bounds for $\gamma_1, \gamma_2, \ldots, \gamma_{10}$ given in the table. Our results indicate that the sequence $\{\gamma_i\}$ converges very quickly to the value 2.30913859... , our best lower bound for γ.

Upper Bounds

For $k = 1,2,\ldots$ we define upper bounds $f^k(m,n)$ for $f(m,n) = \min\{m,n\}$ as follows:

$$(23) \qquad f^k(m,n) = \left\{ \begin{array}{ll} m\ , & \text{if } k < n < m\ , \\ f(m,n)\ , & \text{otherwise.} \end{array} \right\}$$

Hence,

$$(24) \qquad F^k(m,n) = \sum_{m,n=1}^{\infty} f^k(m,n) x^m y^n$$

$$= \frac{xy}{(1-x)^2(1-y)} - \frac{x^2 y}{(1-x)^2} - \cdots - \frac{x^{k+1} y^k}{(1-x)^2}$$

is an upper bound for F; furthermore, note that F^k is separable.

Let B^k denote the solution of (10) with F^k substituted for F. Then,

$$(25) \qquad B^k(x,y) = \frac{xy}{1-xy} + \frac{xyB^k(x,1)}{(1-xy)^2} - \frac{xy}{(1-xy)^2} \sum_{r=1}^{k} x^r y^r B_r^k(x)$$

where

$$B_r^k(x) = \frac{1}{k!} \frac{\partial^r}{\partial s^r} B^k(x,s)\Big|_{s=0} .$$

Now we use (25) to get a syste equations involving $B_1^k, \ldots, B_k^k$. Take the r-th partial derivati e th respect to y at $y = 0$ and divide by $r!$ in (25) to get

$$(26) \qquad B_r^k(x) = x^r + rx^r B^k(x,1) - \sum_{j=1}^{r-1} (r-j)x^r B_j^k(x) ,$$

from which it follows that

$$(27) \qquad B_{r+1}^k(x) = (2x - x^{r+1})B_r^k(x) - x^2 B_{r-1}^k(x) .$$

Setting $B_r^k(x) = P_r(x) + Q_r(x)B^k(x,1)$ for $r = 1,\ldots,k$, it follows that P_r and Q_r also satisfy the difference equation (27). Also, we can substitute $P_r + Q_r B^k$. For B_r in (25) with $y = 1$ and solve for $B^k(x,1)$ in terms of $P_1, Q_1, \ldots, P_k, Q_k$ to obtain

$$(28) \qquad B^k(x,1) = \frac{x - x^2 - \sum_{j=1}^{k} x^{j+1} P_j(x)}{1 - 3x + x^2 + \sum_{j=1}^{k} x^{j+1} Q_j(x)} .$$

Thus, B^k is a rational function whose numerator N_k and denominator D_k we know how to compute because they are defined in terms of $P_1, \ldots, P_k$ and $Q_1, \ldots, Q_k$ which we know how to compute. Let β_k denote the largest real root of $D_k(1/x)$, then we know

$$(29) \qquad \lim_{n\to\infty} \left(\sum_{a=1}^{n} b^k(n,z) \right)^{1/n} = \beta_k \le \gamma ,$$

and $\beta_1 \geq \beta_2 \geq \cdots \geq \gamma$. Thus, we can calculate upper bounds for $\beta_1, \beta_2, \ldots$ to obtain successively better upper bounds for γ .

Using the definitions

$$(30) \qquad D_k = 1 - 3x + x^2 + x^2 Q_1 + \ldots + x^{k+1} Q_k \quad ,$$

$$(31) \qquad Q_{r+1} = (2x - x^{r+1}) Q_r - x^2 Q_{r-1} \qquad (r > 1) \quad ,$$

and $Q_1 = x$, $Q_2 = 2x^2 - x^3$, the polynomials $D_1, D_2, \ldots$ are calculated with relative ease. For example, we found

$$D_1 = 1 - 3x + x^2 + x^3 \quad ,$$

$$D_2 = 1 - 3x + x^2 + x^3 + 2x^5 - x^6 \quad ,$$

$$D_3 = 1 - 3x + x^2 + x^3 + 2x^5 - x^6 + 3x^7 - 2x^8 - 2x^9 + x^{10} \quad .$$

Using a computer, the polynomials $D_1, \ldots, D_{10}$ were calculated via (30), and upper bounds for β_k , the largest real root of $D_k(1/x) = 0$, were computed for $1 \leq k \leq 10$ using the Newton-Raphson method. These upper bounds for β_k are given in the table.

Combining our upper and lower bounds we can conclude that

$$(32) \qquad \gamma = \lim_{n \to \infty} (c(n))^{1/n} = 2.309138\ldots \quad .$$

k	γ_k	β_k
1	1.00000000	2.41421356
2	2.24697960	2.33578290
3	2.30855218	2.31475605
4	2.30913772	2.31023504
5	2.30913859	2.30934711
6	2.30913859	2.30917790
7	2.30913859	2.30914598
8	2.30913859	2.30913998
9	2.30913859	2.30913885
10	2.30913859	2.30913864

Table

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