LINEAR COMBINATIONS OF SETS OF CONSECUTIVE INTEGERS

BY

D. A. KLARNER AND R. RADO

STAN-CS-72-274

MARCH 1972

COMPUTER SCIENCE DEPARTMENT
School of Humanities and Sciences
STANFORD UNIVERSITY

LINEAR COMBINATIONS OF SETS OF CONSECUTIVE INTEGERS

D. A. Klarner
Computer Science Department
Stanford University
Stanford, California 94305

R. Rado */
Department of Mathematics
University of Reading
Reading, England

Abstract

Let $k-1, m_1, \ldots, m_k$ denote non-negative integers, and suppose the greatest common divisor of $m_1, \ldots, m_k$ is 1 . We show that if $S_1, \ldots, S_k$ are sufficiently long blocks of consecutive integers, then the set $m_1 S_1 + \ldots + m_k S_k$ contains a sizable block of consecutive integers. For example; if m and n are relatively prime natural numbers, and u , U , v , V are integers with $U-u \geq n-1$, $V-v \geq m-1$, then the set $m\{u,u+1,\ldots,U\} + n\{v,v+1,\ldots,V\}$ contains the set $\{mu+nv-\sigma(m,n), \ldots, mU+nV-\sigma(m,n)\}$ where $\sigma(m,n) = (m-1)(n-1)$ is the largest number such that $\sigma(m,n)-1$ cannot be expressed in the form $mx+ny$ with x and y non-negative integers.

This research was supported in part by the Office of Naval Research under contract number N-00014-67-A-0112-0057 NR 044-402, and by the National Science Foundation under grant number GJ-992. Reproduction in whole or in part is permitted for any purpose of the United States Government.

*/ Currently Visiting Professor, Faculty of Mathematics, Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario, Canada.

LINEAR COMBINATIONS OF SETS OF CONSECUTIVE INTEGERS

by D. A. Klarner and R. Rado

Let $k-1, m_1, \ldots, m_k$ denote positive integers such that $m_1, \ldots, m_k$ have greatest common divisor 1 , and let t denote an integer. A well-known result in the elementary theory of numbers is that the equation

$$(1) \qquad m_1x_1 + \ldots + m_kx_k = t$$

has infinitely many solutions in integers $x_1, \ldots, x_k$. Furthermore, there exists an integer $\sigma(\bar{m})$ which depends on $\bar{m} = (m_1, \ldots, m_k)$ such that (1) has a solution in non-negative integers $x_1, \ldots, x_k$ for all $t \geq \sigma(\bar{m})$, but no solution of this kind exists when $t = \sigma(\bar{m})-1$. In this note we prove a refinement of this result by showing that a set of consecutive integers can be obtained by allowing the x_i in (1) to range over suitable sets of consecutive integers. For example, every number t with $6 < t < 11$ can be expressed in the form $3x + 4y$ with $0 \leq x \leq 3$, $0 \leq y \leq 2$. Later on we express facts like this by writing

$$(2) \qquad [6,11] \subseteq 3[0,3] + 4[0,2] .$$

The following notation is used: I , N , and P denote the set of all integers, the set of all non-negative integers, and the set of all positive integers respectively. Also, for any pair of elements $i, j \in I$, define $[i,j] = \{x: x \in I,\ i \leq x \leq j\}$; furthermore, given sets $I_1, \ldots, I_k \subseteq I$ together with elements $m_1, \ldots, m_k \in I$, define

(3) $$m_1I_1 + \cdots + m_kI_k = \{m_1x_1 + \cdots + m_kx_k : x_i \in I_i \ (i = 1,\dots,k)\} .$$

For each $k \in P$ and $J \subset I$, let J^k denote the set of all k-dimensional vectors over J ; next, for elements $\bar{x},\bar{y} \in I^k$ with $\bar{x} = (x_1,\dots,x_k)$, $\bar{y} = (y_1,\dots,y_k)$ define the usual dot product $\bar{x}\cdot\bar{y} = x_1y_1 + \cdots + x_ky_k$; finally, define $\bar{x} < \bar{y}$ whenever $x_i < y_i$ for $i = 1,\dots,k$, and define $\bar{x} \leq \bar{y}$ whenever $x_i \leq y_i$ for $i = 1,\dots,k$.

Our main result may be succinctly stated in this notation as follows.

<u>THEOREM 1</u>: Suppose $k-1, m_1,\dots,m_k \in P$ and $m_1,\dots,m_k$ have greatest common divisor 1 ; let $\bar{m} = (m_1,\dots,m_k)$ and $m = \max\{m_1,\dots,m_k\}$; suppose $\bar{u},\bar{v} \in I^k$ satisfy

(4) $$\bar{v}-\bar{u} \geq (m-1,\dots,m-1)$$

(5) $$\bar{m}\cdot(\bar{v}-\bar{u}) > 2(m-1)(m_1 + \cdots + m_k) .$$

Then

(6) $$[\bar{m}\cdot\bar{u} + \sigma(\bar{m}) , \bar{m}\cdot\bar{v} - \sigma(\bar{m})] \subseteq m_1[u_1,v_1] + \cdots + m_k[u_k,v_k] ,$$

where $\bar{u} = (u_1,\dots,u_k)$, $\bar{v} = (v_1,\dots,v_k)$, and $\sigma(\bar{m})$ is the function defined after (1).

Before proving Theorem 1, we shall state and prove a result dealing with the 2-dimensional situation which is sharper than the result provided by taking $k = 2$ in Theorem 1. Furthermore, the proof of Theorem 2 gives some insight for the proof of Theorem 1.

THEO& 2: Suppose $m_1, m_2 \epsilon P$ such that m_1 and m_2 are relatively prime; also, suppose $u_1, u_2, v_1, v_2 \in I$ such that $v_1 - u_1 \geq m_2 - 1$, $v_2 - u_2 \geq m_1 - 1$. Then

$$(7) \qquad [m_1u_1 + m_2u_2 + (m_1-1)(m_2-1), m_1v_1 + m_2v_2 - (m_1-1)(m_2-1)]$$
$$\subseteq m_1[u_1, v_1] + m_2[u_2, v_2] .$$

Proof: It is well-known that $\sigma(m_1, m_2) = (m_1-1)(m_2-1)$, where $\sigma(m_1, m_2)-1$ denotes the largest integer not expressible in the form $m_1x + m_2y$ with $x, y \epsilon N$. Let $\bar{m} = (m_1, m_2)$, $\bar{u} = (u_1, u_2)$, and $v = (v_1, v_2)$, then it follows from the definition of $\sigma(\bar{m})$ that

$$(8) \qquad \bar{m}\cdot\bar{u} + \sigma(\bar{m}) + N \subseteq m_1(u_1+N) + m_2(u_2+N) ,$$

$$(9) \qquad \bar{m}\cdot\bar{v} - \sigma(\bar{m}) - N \subseteq m_1(v_1-N) + m_2(v_2-N) .$$

Hence, the intersection of the sets on the left in (8) and (9) is contained in the intersection of the sets on the right in (8) and (9). That is,

$$(10) \qquad [\bar{m}\cdot\bar{u} + \sigma(\bar{m}), \bar{m}\cdot\bar{v} - \sigma(\bar{m})] \subseteq$$
$$(m_1(u_1+N) + m_2(u_2+N)) \cap (m_1(v_1-N)\ m_2(v_2-N)) .$$

Now we prove a remarkable identity which gives a valid instance of intersection distributing over addition.

$$(11) \qquad (m_1(u_1+N) + m_2(u_2+N)) \cap (m_1(v_1-N) + m_2(v_2-N)) =$$
$$m_1((u_1+N) \cap (v_1-N)) + m_2((u_2+N) \cap (v_2-N)) .$$

Of course, the set on the right in (11) is just

$$(12) \qquad m_1[u_1, v_1] + m_2[u_2, v_2] ,$$

so (10), (11), and (12) combine to imply (7). It remains to prove (11).

Consider the set of points $I \times I$ in the Cartesian plane. The subsets $(u_1+N) \times (u_2+N)$ and $(v_1-N) \times (v_2-N)$ of $I \times I$ lie in upper and lower quadrants of the plane whose intersection contains the set $[u_1,v_1] \times [u_2,v_2]$. This situation is illustrated in Figure 1. We want to study the linear form m_1x+m_2y evaluated over all points $(x,y) \in I \times I$; in particular, we are interested in points which have equal evaluations. Given an element $h \in I$, the set L_h of all points $(x,y) \in I \times I$ such that $m_1x+m_2y = h$ is situated on a unique line having slope $-m_1/m_2$. Also, it is easy to see that if $(x',y') \in (I \times I) \cap L_h$, then $L_h = \{(x'+jm_2, y'-jm_1): j \in I\}$.

To prove (11), note that the set on the right is contained in the set on the left; suppose the reverse is not true. From this assumption we shall deduce a contradiction. Under this assumption it follows that there exists an $h \in I$ such that L_h has points in common with both

$$U = ((u_1+N) \times (u_2+N)) \setminus ([u_1,v_1] \times [u_2,v_2])$$

and

$$V = ((v_1-N) \times (v_2-N)) \setminus ([u_1,v_1] \times [u_2,v_2]) ,$$

but L_h has no point in common with

$$B = [u_1,v_1] \times [u_2,v_2] .$$

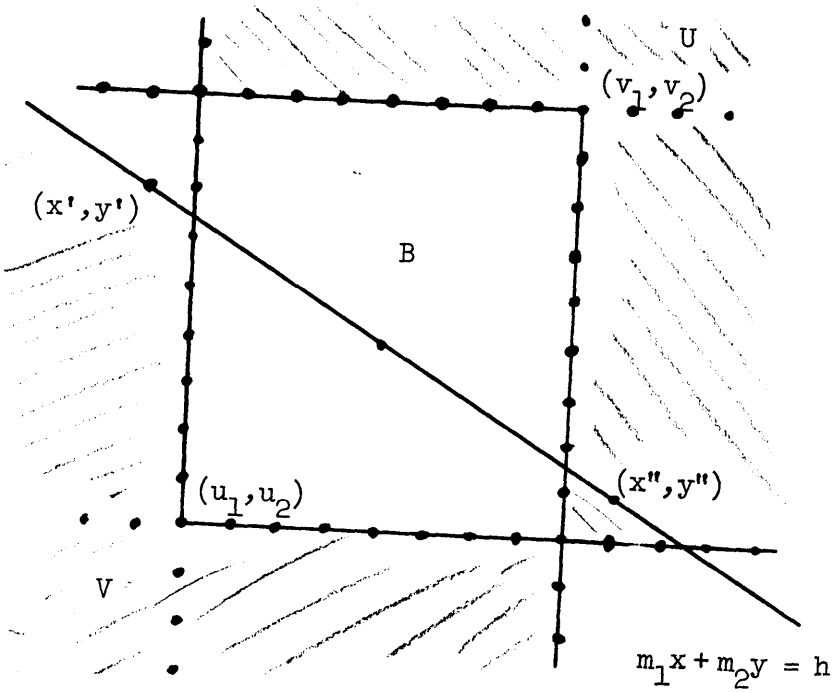

Figure 1. The set of points $(u_1+N) \times (u_2+N)$ lies in the quadrant above and to the right of the point (u_1,u_2), the set of points $(v_1-N) \times (v_2-N)$ lies in the quadrant below and to the left of the point (v_1,v_2), and the set of points $[u_1,v_1] \times [u_2,v_2]$ lies in the box.

Suppose $(x',y') \in L_h \cap U$ and $(x'',y'') \in L_h \cap V$; since $(x',y') \notin B$, either $x' < u_1$ or $y' > v_2$. If $x' < u_1$, then $x'' > v_1$ because $(x',y'),(x'',y'') \in L_h$ and $(x'',y'') \notin B$. In this case we suppose (x',y') has been selected from $L_h \cap U$ so that x' is maximal, and (x'',y'') has been selected from $L_h \cap V$ so that x'' is minimal. Since $(x',y'),(x'',y'') \in L_h$, and $L_h \cap B = \emptyset$, we must have $x''-x' = m_2$. But, $x' < u_1$ and $x'' > v_1$ implies $x'+1 \leq u_1$ and $x''-1 \geq v_1$; hence, $m_2-2 = x''-x'-2 \geq v_1-u_1$, contradicting the hypothesis $v_1-u_1 \geq m_2-1$. In the case $y' > v_2$, it follows that $y'' < u_2$. This time the points (x',y') and (x'',y'') are selected so that y' is minimal and y'' is maximal. The argument goes just as before; we must have $y'-y'' = m_1$ which leads to the contradiction $v_2-u_2 \leq m_1-2$. This completes the proof of Theorem 2.

Now we prove Theorem 1. To do this, we prove an identity having the form of (11), but subject to the conditions (4) and (5).

<u>LEMMA</u>. If k-dimensional vectors $\bar{m}$, $\bar{u}$, and $\bar{v}$ satisfy the hypothesis of Theorem 1, then

$$(13) \qquad \sum_{i=1}^{k} m_i(u_i+N) \cap \sum_{i=1}^{k} m_i(v_i-N) = \sum_{i=1}^{k} m_i((u_i+N) \cap (v_i-N))$$

Theorem 1 is an immediate consequence of the Lemma; its application is the justification of the penultimate equality in the following string of formulas.

(14) $$[\bar{m}\cdot\bar{u}+\sigma(\bar{m}), \bar{m}\cdot\bar{v}-\sigma(\bar{m})] =$$

$$(\bar{m}\cdot\bar{u}+\sigma(\bar{m})+N) \cap (\bar{m}\cdot\bar{v}-\sigma(\bar{m})-N) \subset$$

$$\sum_{i=1}^{k} m_i(u_i+N) \cap \sum_{i=1}^{k} m_i(v_i-N) =$$

$$\sum_{i=1}^{k} m_i((u_i+N) \cap (v_i-N)) = \sum_{i=1}^{k} m_i[u_i,v_i] .$$

To prove Theorem 1 completely, it remains to prove the Lemma. For each $i\epsilon I$, let $L_i = \{ii: \bar{x}\epsilon I^k;\ \bar{m}\cdot\bar{x} = i\}$, and suppose the Lemma is false. Then there exists $h\epsilon I$ such that $L_h \cap U$, $L_h \cap V \neq \emptyset$, but $L_h \cap B = \emptyset$ where

$$U = \{\bar{x}: \bar{x}\epsilon I^k, \bar{x} \geq \bar{u}\} \setminus B$$

$$V = \{\bar{x}: \bar{x}\epsilon I^k, \bar{x} \leq \bar{v}\} \setminus B$$

$$B = [u_1,v_1] \times \cdots \times [u_k,v_k] .$$

Suppose $\bar{x}'\epsilon U$ is selected so that

(15) $$\sum_{i=1}^{k} \max\{v_i, x_i'\}$$

is minimal, where $\bar{x}' = (x_1',\ldots,x_k')$. Since $\bar{x}'\notin B$, there exists $r\epsilon[1,k]$ such that $x_r' > v_r$. Furthermore, there exists $s\epsilon[1,k]$ such 'that $x_s' \leq v_s$ since otherwise $\bar{x}' > \bar{v}$, which implies $h = \bar{m}\cdot\bar{x}' > \bar{m}\cdot x$ for all $\bar{x} < \bar{v}$, contradicting the assumption $L_h \cap V \neq \emptyset$. Of course, $r \neq s$, sowehave

(16) $$h = \sum_{\substack{i=1\\ i\neq r,s}}^{k} m_i x_i' + m_r(x_r' - m_s) + m_s(x_s' + m_r) ;$$

$$(17)\qquad x'_r - m_s - u_r \geq (v_r+1) - m_s - u_r = (v_r - u_r) - m_s + 1 >$$

$$(v_r - u_r) - m + 1 > 0 \quad .$$

Hence, by the minimality assumption made in (15),

$$(18)\qquad \max\{v_r , x'_r - m_s\} + \max\{v_s , x'_s + m_r\} \geq \max\{v_r, x'_r\} + \max\{v_s, x'_s\} \quad .$$

Hence,

$$(19)\qquad \max\{v_s , x'_s + m_r\} > \max\{v_s, x'_s\} = v_s \quad ;$$

$$x'_s + m_r > v_s \quad ;$$

$$x'_s > v_s - m_r \geq v_s - m \quad .$$

This implies

$$(20)\qquad \bar{x}' > \bar{v} - (m,\ldots,m) \quad .$$

Suppose $x'' \epsilon V$ is selected so that

$$(21)\qquad \sum_{i=1}^{k} \min\{u_i, x''_i\}$$

is maximal where $\bar{x}'' = (x'_1,\ldots,x'_k)$. Now an argument running parallel to (15)-(21) can be given to show that

$$(22)\qquad \bar{x}'' < \bar{u} + (m,\ldots,m) \quad .$$

Together (20) and (22) imply

$$(23)\qquad 0 = \bar{m}\cdot\bar{x}' - \bar{m}\cdot\bar{x}'' \geq \sum_{i=1}^{k} m_i((v_i - m + 1) - (u_i + m - 1))$$

$$= \bar{m}\cdot(\bar{v}-\bar{u}) - 2(m-1) \sum_{i=1}^{k} m_i \quad .$$

But (5) implies

$$(24) \qquad \bar{m}\cdot(\bar{v}-\bar{u}) - 2(m-1)\sum_{i=1}^{k} m_i > 0 \quad ,$$

so (23) provides the required contradiction, and we conclude that the Lemma is true.

The results proved in this paper arose in connection with our investigation [1] of the smallest set $\langle \bar{m}\cdot\bar{x} : 1\rangle \subset P$ containing 1 which is closed under the operation $\bar{m}\cdot\bar{x}$ where $\bar{m} = (m_1, \ldots, m_k)$ is a given k-tuple of relatively prime positive integers.

References

[1] D. A. Klarner and R. Rado, "Arithmetic Properties of Certain Recursively Defined Sets," to appear.