RECURRENCE RELATIONS BASED ON MINIMIZATION

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STAN-CS-72-248
DECEMBER, 1971

COMPUTER SCIENCE DEPARTMENT
School of Humanities and Sciences
STANFORD UNIVERSITY

Unclassified

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Stanford University	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

RECURRENCE RELATIONS BASED ON MINIMIZATION

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Technical - December 1971

5. AUTHOR(S) (First name, middle initial, last name)

Michael L. Fredman and Donald E. Knuth

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
December 1971	35	18

8a. CONTRACT OR GRANT NO ONR N-00014-67-A-0112 NR 044-402 NSF GJ-992 b. PROJECT NO. c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) STAN-CS-71-248 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. DISTRIBUTION STATEMENT

Approved for public release; distribution unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Office of Naval Research Washington, D. C.

13. ABSTRACT

This paper investigates solutions of the general recurrence

$$M(0) = g(0) \quad , \quad M(n+1) = g(n+1) + \min_{0 \le k \le n}(\alpha M(k) + \beta M(n-k))$$

for various choices of α , β , and $g(n)$. In a large number of cases it is possible to prove that $M(n)$ is a convex function whose values can be computed much more efficiently than would be suggested by the defining recurrence. The asymptotic behavior of $M(n)$ can be deduced using combinatorial methods in conjunction with analytic techniques. In some cases there are strong connections between $M(n)$ and the function $H(x)$ defined by

$$H(x) = 1 \ \text{for} \ x < 1 \ , \ H(x) = H((x-1)/\alpha) + H((x-1)/\beta) \ \text{for} \ x \ge 1 \ .$$

Special cases of these recurrences lead to a surprising number of interesting problems involving both discrete and continuous mathematics.

DD FORM 1 NOV 65 1473 (PAGE 1)

S/N 0101-807-6801

Unclassified

Security Classification

Unclassified
Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Discrete dynamic programming						
convex functions						
binary trees						
gapless partitions						
asymptotic methods						
generalized Fibonacci numbers						
Dirichlet integrals						

Unclassified
Security Classification

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Keywords and phrases: Discrete dynamic programming, convex functions, binary trees, gapless partitions, asymptotic methods, generalized Fibonacci numbers, Dirichlet integrals.

AMS Subject Classifications: Primary 10A35, 39A10, 41A60; secondary 05A15, 40E05, 49B05.

This research was supported by the National Science Foundation under grant number GJ-992, and the Office of Naval Research under grant number N-00014-67-A-0112-0057 NR 044-402. Reproduction in whole or in part is permitted for any purpose of the United States Government.

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Recurrence Relations Based on Minimization

Let α and β be positive real constants, and let $g(n)$ be a real-valued function over the nonnegative integers. Consider the new function $M_{g\alpha\beta}(n)$ over the nonnegative integers, defined as follows:

$$M_{g\alpha\beta}(0) = g(0) \ ,$$

$$M_{g\alpha\beta}(n+1) = g(n+1) + \min_{0 \leq k \leq n} (\alpha M_{g\alpha\beta}(k) + \beta M_{g\alpha\beta}(n-k)) \ . \qquad (0.1)$$

We shall occasionally write $M(n)$ instead of $M_{g\alpha\beta}(n)$. Functions of this type occur in discrete dynamic programming situations, where it is important to study the behavior of $M_{g\alpha\beta}(n)$ for large n.

The purpose of this paper is to introduce some techniques which are useful in the investigation of $M_{g\alpha\beta}(n)$, and in some cases to obtain ways of computing $M_{g\alpha\beta}(n)$ with much less work than the above definition implies. Particular attention is paid to the cases $g(n) = \delta_{n0}$, $g(n) = 1$, $g(n) = n$, and $g(n) = n^2$, where asymptotic formulas are derived.

1. A convexity theorem

A real valued function $g(n)$ over the nonnegative integers is called convex if its second difference is nonnegative, i.e., if $g(n+2)-g(n+1) \geq g(n+1)-g(n)$ for all $n \geq 0$. The following theorem shows that a large class of $M_{g\alpha\beta}$ functions is convex, and it characterizes the function $D_{g\alpha\beta}(n) = M_{g\alpha\beta}(n+1)-M_{g\alpha\beta}(n)$ in this class.

Theorem 1. *Let* $g(n)$ *be a function which satisfies the following conditions:*

a) $g(n+2)-g(n+1) \geq g(n+1)-g(n)$ for all $n \geq 1$;

b) $g(2)-g(1)+\min(\alpha d, \beta d) \geq d$, where

$$d = g(1) - g(0) + (\alpha+\beta) g(0) \quad . \tag{1.1}$$

There is a unique function $D(n)$ *satisfying the following three properties:*

(i) $D(0) \leq D(1)$;

(ii) $D(0) = d$ (cf. (1.1));

(iii) $D(n) = g(n+1)-g(n)+F(n)$, for $n \geq 1$, *where the infinite sequence* $\langle F(1), F(2), F(3), \ldots \rangle$ *is the result of sorting the sequence* $\langle \alpha D(0), \beta D(0), \alpha D(1), \beta D(1), \ldots \rangle$ *into nondecreasing order.*

(Sometimes the infinite sequence $\langle F(1), F(2), \ldots \rangle$ fails to include all the elements of $\langle \alpha D(0), \beta D(0), \ldots \rangle$, e.g. when $\alpha D(n) < \beta D(0)$ for all n .)

This function D *is nondecreasing, and we have*

$$M_{g\alpha\beta}(n) = g(0) + \sum_{0 \leq j < n} D(j) \quad . \tag{1.2}$$

Thus, $M_{g\alpha\beta}$ *is convex.*

Proof: Consider the function $D^*(n)$ defined by the rules

$$D^*(0) = d$$
$$D^*(n) = g(n+1)-g(n)+F_{nn} \quad , \quad \text{for } n \geq 1 \, , \tag{1.3}$$

where $\langle F_{n1}, F_{n2}, \ldots, F_{n,2n} \rangle$ is the sequence obtained when $\langle \alpha D^*(0), \beta D^*(0), \ldots, \alpha D^*(n-1), \beta D^*(n-1) \rangle$ is sorted into nondecreasing order. We shall prove by induction on $n \geq 1$ that $D^*(n) \geq D^*(n-1)$ and that $F_{n+1,n+1} \geq F_{n,n}$. When $n = 1$, we have $D^*(1) = g(2)-g(1)+\min(\alpha d, \beta d) \geq D^*(0)$ by condition (b). Hence

$F_{22} \geq F_{11} = \min(\alpha d, \beta d)$. For $n \geq 2$, the relation $F_{nn} \geq F_{n-1,n-1}$ together with condition (a) shows that $D^*(n) \geq D^*(n-1)$. Consequently the first n elements of $\langle F_{n+1,1}, F_{n+1,2}, \ldots \rangle$ are the same as those of $\langle F_{n1}, F_{n2}, \ldots \rangle$, and we have $F_{n+1,n+1} \geq F_{n,n}$.

This argument shows that $F_{n,n} = F_{m,n}$ for all $m \geq n$, hence $\langle F_{11}, F_{22}, F_{33}, \ldots \rangle$ is the result of sorting the sequence $\langle \alpha D^*(0), \beta D^*(0), \alpha D^*(1), \beta D^*(1), \ldots \rangle$ into nondecreasing order. Hence D^* satisfies the conditions (i), (ii), (iii). Conversely if D is any function satisfying (i), (ii), (iii) we have $D(0) \leq D(1) \leq D(2) \leq \cdots$ by (i) and condition (a), hence D must satisfy the recurrence relations defining $D^*(n)$. This proves the existence and uniqueness of $D(n)$.

Finally we need to prove (1.2), for $n \geq 1$. By the definition of $\langle F(1), F(2), F(3), \ldots \rangle$, we have

$$\sum_{1 \leq j \leq n} F(j) \leq \alpha \sum_{0 \leq j < k} D(j) + \beta \sum_{0 \leq j < n-k} D(j) ,$$

for all $0 \leq k \leq n$, and equality holds for some k . Thus,

$$g(0) + \sum_{0 \leq j \leq n} D(j) = g(n+1) + (\alpha+\beta) g(0) + \sum_{1 \leq j \leq n} F(j)$$

$$= g(n+1) + \min_{0 \leq k \leq n} \Bigl(\alpha\bigl(g(0) + \sum_{0 \leq j < k} D(j)\bigr) + \beta\bigl(g(0) + \sum_{0 \leq j < n-k} D(j)\bigr) \Bigr) .$$

□

It is interesting to note that condition (i), or something similar, is necessary for the validity of this theorem. For example, assume that $\alpha = \beta = 1$ and that $g(n) = 1$ for all n . Then the two functions $D_1(n) = 2$ and $D_2(n) = 2\delta_{n0}$ both satisfy conditions (ii) and (iii)! This accounts for the somewhat complicated formula in condition (b).

Note that we can compute the M function using the following simple algorithm, whenever g(n) satisfies the hypotheses of Theorem 1:

```
begin
      integer j,k,n;
      real M,F;
      array D[0:N];
   j := k := 0;  D[0] := g(1)-g(0)+(α+β)×g(0);
   for n := 1 step 1 until N do
   begin if α×D[j] ≤ β×D[k] then
            begin F := α×D[j]; j := j+1 end
         else begin F := β×D[k]; k := k+1 end;
      D[n] := g(n+1)-g(n)+F;
   end  computation of D;
   M := g(0);
   for n := 0 step 1 until N do
   begin print ('n= ', n, '; D[n]= ', D[n], '; M[n]= ', M);
      M := M+D[n];
   end printing the table of D and M.
```

This algorithm takes only $O(N)$ steps to compute $M[0],M[1],\ldots,M[N]$, instead of the $O(N^2)$ steps which are implied by the original definition of $M_{g\alpha\beta}(n)$ in (0.1).

Theorem 1 also has a useful corollary when α and β are equal:

Corollary. *Let* $\alpha = \beta$ *and let* $g(n)$ *be as in Theorem 1. Then*

$$M_{g\alpha\alpha}(n) = g(n) + \alpha\left(M_{g\alpha\alpha}\left(\left\lfloor \frac{n-1}{2} \right\rfloor\right) + M_{g\alpha\alpha}\left(\left\lceil \frac{n-1}{2} \right\rceil\right)\right) \quad \text{for all } n \geq 1 . \qquad (1.4)$$

(Here $\lfloor x \rfloor$, $\lceil x \rceil$ respectively denote the greatest integer $\leq x$ and the least integer $\geq x$.)

Proof: By Theorem 1 with $\beta = \alpha$, M is convex. It is easy to prove for any convex function M that the minimum value of $M(k) + M(n-k)$ occurs for $k = \lfloor n/2 \rfloor$. (Note further that $\langle F(1), F(2), F(3), F(4), \ldots \rangle = \langle \alpha D(0), \alpha D(0), \alpha D(1), \alpha D(1), \ldots \rangle$ in this case.)

□

2. The case $g(n) = n$: "optimal trees"

When $g(n) = n$, so that $D(0) = 1$ and $D(n) = 1+F(n)$ in Theorem 1, we are soon led to an interpretation of $M_{g\alpha\beta}(n)$ in terms of binary trees. In this section we shall develop this tree relationship in an independent manner, without explicitly using the result of Theorem 1. Our general plan is to define a weighting function for the nodes of a binary tree; $M_{g\alpha\beta}(n)$ will turn out to be the minimum total weight of any binary tree with n nodes. (See [11] for an introduction to the well-known properties of binary trees.)

A binary tree T is, by definition, either empty or it consists of a left subtree $\ell(T)$, a right subtree $r(T)$, and an apex or root node $a(T)$; $\ell(T)$ and $r(T)$ are themselves binary trees. Let Λ denote the empty binary tree, and let $|T|$ be the number of nodes of T. Thus,

$$|T| = \begin{cases} 0 & , \text{ if } T = \Lambda ; \\ 1 + |\ell(T)| + |r(T)| , & \text{ if } T \neq \Lambda . \end{cases} \tag{2.1}$$

Now consider the function

$$\mathcal{M}(T) = \begin{cases} 0 & , \text{ if } T = \Lambda ; \\ |T| + \alpha\, \mathcal{M}(\ell(T)) + \beta\, \mathcal{M}(r(T)) & , \text{ if } T \neq \Lambda , \end{cases} \tag{2.2}$$

and let

$$M(n) = \min_{T:\ |T|=n} \mathcal{M}(T) \quad . \tag{2.3}$$

We shall say T is "optimal" if $\mathcal{M}(T) = M(|T|)$. It is easy to see that the "principle of optimality" of dynamic programming is satisfied, in the sense that all subtrees of an optimal tree must be optimal. Consequently for $n > 0$ we have

$$M(n) = n + \min_{0 \le k < n} (\alpha M(k) + \beta M(n-1-k)) \quad ,$$

i.e., $M(n) = M_{g\alpha\beta}(n)$.

Another way to view the situation is to consider finite strings (i.e., sequences) of the letters L and R . If σ is such a string, define $w(\sigma)$ by the following rules:

$$w(\varepsilon) = 1 \quad ; \quad w(L\sigma) = 1 + \alpha w(\sigma) \quad ; \quad w(R\sigma) = 1 + \beta w(\sigma) \quad . \tag{2.4}$$

Here ε denotes the empty string. As an example of this definition, $w(LRRLL) = 1 + \alpha + \alpha\beta + \alpha\beta^2 + \alpha^2\beta^2 + \alpha^3\beta^2$.

Any node in a binary tree may be uniquely identified by a sequence of L's and R's [7]: we denote $a(T)$ by ε , and denote the nodes of $\ell(T)$ and $r(T)$ by placing an L or R respectively before the denotations in $\ell(T)$, $r(T)$. Thus if $\mathcal{J}(T)$ is the set of all such strings, we have

$$\mathcal{J}(T) = \begin{cases} \emptyset & , \text{ if } T = \Lambda \ ; \\ \{\varepsilon\} \cup L\,\mathcal{J}(\ell(T)) \cup R\,\mathcal{J}(r(T)) & , \text{ if } T \neq \Lambda \ . \end{cases}$$

It is easy to see that a set of strings S is equal to $\mathcal{J}(T)$ for some T if and only if

$$\sigma L \in S \text{ or } \sigma R \in S \text{ implies } \sigma \in S \ . \tag{2.5}$$

Furthermore $\mathcal{M}$ is a "total weight" function, in the sense that

$$\mathcal{M}(T) = \sum_{\sigma \in \mathcal{N}(T)} w(\sigma) \quad . \tag{2.6}$$

This is the basic relation we shall use; it is easily verified by induction.

Now consider a sequence of strings $\langle \sigma_1, \sigma_2, \sigma_3, \ldots \rangle$ such that, for each n, $w(\sigma_n)$ has minimum weight among all strings not in $\{\sigma_1, \ldots, \sigma_{n-1}\}$. Thus, $\sigma_1 = \epsilon$; $\sigma_2 = L$ if $\alpha < \beta$, $\sigma_2 = R$ if $\alpha > \beta$. (For some choices of α and β, e.g. $\alpha = 1/3$ and $\beta = 2/3$, there are infinitely many strings which will never appear in the sequence.) For each n, the set $S_n = \{\sigma_1, \ldots, \sigma_n\}$ defines an optimal binary tree; this follows from (2.5), because $w(\sigma L)$ and $w(\sigma R)$ are always greater than $w(\sigma)$. Consequently

$$M_{g\alpha\beta}(n) = \sum_{1 \leq k \leq n} w(\sigma_k) \quad . \tag{2.7}$$

This explicit interpretation of $M_{g\alpha\beta}$ is essentially that of Theorem 1, since $\langle D(0), D(1), \ldots \rangle$ is precisely the sequence $\langle w(\sigma_1), w(\sigma_2), \ldots \rangle$.

As a simple application of these ideas, we can derive an asymptotic formula.

<u>Theorem 2</u>. <u>Let</u> $g(n) = n$, $0 < \alpha \leq \beta$, <u>and</u> $\alpha < 1$. <u>Then</u>

$$M_{g\alpha\beta}(n) \sim \frac{n}{1-\alpha} \quad . \tag{2.8}$$

<u>Proof</u>: If σ is a string of length $\geq m$, $w(\sigma) \geq w(L^m) = 1 + \alpha + \ldots + \alpha^m = (1-\alpha^{m+1})/(1-\alpha)$. There are only finitely many strings of length $< m$, hence $\liminf M(n)/n \geq (1-\alpha^{m+1})/(1-\alpha)$ for all m. On the other hand, $\limsup M(n)/n \leq 1/(1-\alpha)$, since the sequence of strings $\epsilon, L, L^2, L^3, \ldots$ gives an upper bound. □

3. <u>The case</u> $g(n) = n$: <u>asymptotic results when</u> $\min(\alpha,\beta) = 1$

Theorem 2 shows how M grows when $\min(\alpha,\beta) < 1$. When $\alpha = \beta = 1$ we have $w(\sigma) = m+1$ for all strings σ of length m, hence we can obtain the well-known explicit formula

$$M_{g11}(n) = \sum_{1 \le k \le n} \lceil \log_2(k+1) \rceil = (n+1) \lceil \log_2(n+1) \rceil - 2^{\lceil \log_2(n+1) \rceil} + 1$$

$$= n \log_2 n + O(n) \quad . \tag{3.1}$$

When $\alpha = 1$ and $\beta > 1$, the problem of estimating $M_{g\alpha\beta}(n)$ is considerably more difficult. In this case the weight function $w(\sigma)$ is related to partitions into powers of β ; for example,

$$w(LRRLL) = 1 + 1 + \beta + \beta^2 + \beta^2 + \beta^2 \quad .$$

The weights take the form of polynomials with nonnegative coefficients,

$$a_0 + a_1\beta + a_2\beta^2 + \ldots + a_k\beta^k \quad , \tag{3.2}$$

such that there are no "gaps":

$$a_j > 0 \Rightarrow a_{j-1} > 0 \ . \tag{3.3}$$

An expression of the form (3.2) may be called a <u>partition</u> into powers of β ; if condition (3.3) is also satisfied we shall call it a <u>gapless partition</u>. It is convenient to regard the case $a_0 = a_1 = \ldots = 0$ as a gapless partition, even though it is not the weight of any string σ ; the nonzero gapless partitions are in one-to-one correspondence with strings of L's and R's, since (3.2) is the weight of $L^{a_0-1} R L^{a_1-1} R \ldots R L^{a_k-1}$.

Let $P(x)$ denote the number of partitions into powers of β whose value is $\le x$, and let $H(x)$ be the corresponding number of gapless partitions. Thus, $H(x)$ is the number of strings of weight $\leq x$, plus one. We have $P(x) = H(x) = 1$ for $0 \leq x < 1$, and it is not difficult to deduce the following recurrence relations for $x \geq 1$:

$$P(x) = P(x-1) + P(x/\beta) \quad ; \tag{3.4}$$

$$H(x) = H(x-1) + H((x-1)/\beta) \quad . \tag{3.5}$$

As a consequence, we have the following relation between partitions and gapless partitions:

<u>Lemma 3.1</u>. $c_1^{-1}P(x+1/(\beta-1)) \leq H(x) \leq c_2^{-1}P(x+1/(\beta-1))$, <u>where</u> $c_1 = P(1+1/(\beta-1)-0)$ <u>and</u> $c_2 = P(1/(\beta-1))$.

<u>Proof</u>: Let $H_1(x) = P(x+1/(\beta-1))$. For $x \geq 1$ we have $H_1(x) = P(x-1+1/(\beta-1)) + P((x-1)/\beta+1/(\beta-1)) = H_1(x-1) + H_1((x-1)/\beta)$, and for $0 \leq x < 1$ we have $c_2 \leq H_1(x) \leq c_1$. Thus $c_1^{-1}H_1(x) \leq H(x) \leq c_2^{-1}H_1(x)$ for all x, by induction on $\lfloor x \rfloor$. □

When $\beta = 2$, we have $c_1 = c_2 = 2$, so the above lemma shows that the number of gapless partitions of n into powers of 2 is exactly half the number of ordinary partitions of $n+1$ into powers of 2, for all positive integers n. A combinatorial proof of this result is also possible: The number of ordinary partitions (3.2) of n in which $a_k = 1$ is the same as the number with $a_k > 1$, under the correspondence $(a_0, a_1, \ldots, a_{k-1}, 1) \leftrightarrow (a_0, a_1, \ldots, a_{k-1}+2)$. The number of ordinary partitions of n in which $a_k = 1$ is the same as the number of gapless partitions of $n-1$, under the correspondence $(a_0, a_1, \ldots, a_{k-1}, 1) \leftrightarrow (a_0+1, a_1+1, \ldots, a_{k-1}+1)$.

The H function has a comparatively simple relation to M, namely

$$M(H(x)-1) = \int_0^x t\,dH(t) = xH(x) - \int_0^x H(t)dt \quad , \qquad (3.6)$$

since $M(H(x)-1)$ is the sum of all gapless partitions whose value is $\leq x$ (cf. (2.7)). Therefore we can use known results about partitions into powers of β in order to deduce the asymptotic behavior of M :

<u>Theorem 3</u>. <u>When</u> $\beta > 1$ <u>and</u> $g(n) = n$, <u>we</u> <u>have</u>

$$M_{g1\beta} \sim \frac{1}{e}\sqrt{\frac{2 \ln n}{\beta \ln \beta}}\; n^{1+\sqrt{2 \ln \beta/\ln n}} \qquad (3.7)$$

<u>Proof</u>: N. G. de Bruijn [3] has proved that

$$\ln P(x) = \ln \beta\left(\frac{y^2}{2} + y\right) + \left(\frac{1}{\ln \beta} - \frac{1}{2}\right)\ln x + p(y) + O\left(\frac{(\log \log x)^2}{\log x}\right), \qquad (3.8)$$

where $y = \log_\beta x - \log_\beta \log_\beta x$, and where p is a rather horrible-looking function of period 1 , namely

$$p(y) = (\gamma_2 - \tfrac{1}{2}\gamma^2 + \tfrac{1}{12}\pi^2) / \ln \beta + \tfrac{1}{12}\ln \beta - \tfrac{1}{2}\ln 2\pi$$
$$+ \sum_{k\neq 0} \Gamma\left(\frac{2\pi i k}{\ln \beta}\right) \zeta\left(1 + \frac{2\pi i k}{\ln \beta}\right) e^{2\pi i k y} / \ln \beta \quad , \qquad (3.9)$$

where $z\zeta(z+1) = 1 + \gamma z + \gamma_2 z^2 + \ldots$.

Now we wish to show that the integral $\int_0^x H(t)dt$ in (3.6) is small with respect to the other term $xH(x)$. We have

$$\int_0^x H(t)dt = \int_0^x (H(\beta t+1) - H(\beta t))dt$$

$$= \beta^{-1}\left(\int_{\beta x}^{\beta x+1} H(u)du - \int_0^1 H(u)du\right) = O(H(\beta x)) \quad .$$

By (3.8) and Lemma 4.1, $\ln(H(\beta x)/H(x)) = y \ln \beta + O(1)$, hence

$$H(\beta x) = O(xH(x)/\log x) \quad . \tag{3.10}$$

If we set $n = H(x)-1$ and $M(n) = ne^{f(n)}$, we now have

$$M(n) = xn + O(xn/\log x) \quad , \tag{3.11}$$

$$f(n) = \ln x + O(1/\log x) \quad , \tag{3.12}$$

and it remains to express $\ln x$ in terms of n .

We have $\ln x = y \ln \beta + \ln y + O(\log \log x/\log x)$; hence by (3.8) and Lemma 4.1,

$$\ln n = \frac{\ln \beta}{2} y^2 + (1 + \frac{\ln \beta}{2})y + (\frac{1}{\ln \beta} - \frac{1}{2}) \ln y + O(1) \quad .$$

Consequently

$$y = \sqrt{2 \log_\beta n} - \frac{1}{2} - \frac{1}{\ln \beta} + O(\frac{\log \log n}{\sqrt{\log n}}) \quad ,$$

and (3.7) follows immediately for those values of n having the special form $H(x)-1$. In general suppose that $H(x-0)-1 = n_0 < n \le n_1 = H(x)-1$. Then $n_1-n_0 \le H(x) - H(x-1) = H((x-1)/\beta) = O(H(x) \log x/x) = o(n_1)$, hence $n_0/n_1 \to 1$ as $n \to \infty$; by (3.11), $M(n_0)/M(n_1) \to 1$. □

The above proof can be extended to obtain slightly more information than is stated in Theorem 3; we could evaluate $f(n)$ to within $O(1/\sqrt{\log n})$. But the complicated form of (3.9) shows that it is inherently very difficult to go any further than this.

Before moving to the next topic, let us digress for a moment to summarize the interesting history of the present case. Euler gave the generating function for partitions into powers of 2 in his famous paper on partitions [5]. A. Cayley [1] proved that the number of sequences $a_1, a_2, \ldots, a_k$ such that $a_1 = 1$ and $1 \le a_{i+1} \le 2a_i$ is equal to the number of partitions of $2^k - 1$ into powers of 2 ; he proved this using the corresponding generating function. Binary partitions were independently studied by Tanturri [16]. The behavior of the generating function in the neighborhood of unity was investigated about 1923 by C. L. Siegel, in unpublished work. P. Erdös [4] found the leading term of (3.8), and K. Mahler [13] found the other terms except with $O(1)$ instead of the periodic function $p(y)$, when β is an integer. N. G. de Bruijn [3] obtained (3.8) for all $\beta > 1$, and his work was further generalized by W. B. Pennington [14]. The connection between binary partitions and the M_{g12} function was pointed out by Knuth [10], who gave an elementary derivation of the leading term in (3.8) when $\beta = 2$. Heller [8] found the leading terms of (3.8) using a different approach. Arithmetic properties of β-ary partitions have been studied by Churchhouse [2] and Rödseth [15].

4. <u>The case</u> $g(n) = n$: <u>asymptotic results when</u> $\min(\alpha,\beta) > 1$.

When $\alpha = \beta$, the weight of any string σ is simply $1 + \alpha + \ldots + \alpha^{|\sigma|}$; so it is easy to obtain an "explicit" formula for $M_{g\alpha\beta}$ when $g(n) = n$ and $\alpha = \beta$:

$$M_{g\alpha\alpha}(2^m + k - 1) = 2^m(1 + \alpha + \ldots + \alpha^{m-1}) - (1 + 2\alpha + \ldots + 2^{m-1}\alpha^{m-1}) + k(1 + \alpha + \ldots + \alpha^m) , \quad \text{for } 0 \le k \le 2^m . \quad (4.1)$$

It follows that for $0 \leq \theta \leq 1$ and $\alpha > 1$,

$$\lim_{m \to \infty} \frac{M_{g\alpha\alpha}((1+\theta)2^m)}{2^m \alpha^m} = \frac{1}{\alpha-1} - \frac{1}{2\alpha-1} + \frac{\theta\alpha}{\alpha-1} . \quad (4.2)$$

Replacing $(1+\theta)2^m$ by n, it follows that

$$M_{g\alpha\alpha}(n) \sim c(\theta)\ n^{1+\log_2 \alpha}$$

where $\theta = 2^{(\log_2 n) \bmod 1} - 1$ and

$$c(\theta) = \left(\frac{1+\theta\alpha}{\alpha-1} - \frac{1}{2\alpha-1}\right)(1+\theta)^{-(1+\log_2 \alpha)}$$

is a periodic function of $\log_2 n$. For example, when $\alpha = \beta = 2$, the asymptotic form of $M_{g22}(n)$ varies between $\frac{2}{3} n^2$ (when $n \approx 2^m$) and $\frac{3}{4} n^2$ (when $n \approx \frac{4}{3} 2^m$).

We shall see that such behavior is typical of the case $\min(\alpha,\beta) > 1$. If we define the constant γ by the relation

$$\alpha^{-\gamma} + \beta^{-\gamma} = 1 , \quad (4.3)$$

we will find that $M_{g\alpha\beta}(n)$ grows approximately as $n^{1+1/\gamma}$. When $\log \alpha/\log \beta$ is <u>irrational</u>, it turns out that $M_{g\alpha\beta}(n)/n^{1+1/\gamma}$ actually approaches a limit as $n \to \infty$. On the other hand in many cases when $\log \alpha/\log \beta$ is rational, $M_{g\alpha\beta}(n)/n^{1+1/\gamma}$ oscillates between two different limits, as in the case $\alpha = \beta$.

We shall begin our analysis of the general case $g(n) = n$, $\min(\alpha,\beta) > 1$ by generalizing the H function used in Section 3. Let $h(x)$ be the number of strings σ whose weight $w(\sigma)$ is $\leq x$, and let

$$H(x) = h(x) + 1 . \quad (4.4)$$

We have $H(x) = 1$ for $0 \leq x < 1$, and for $x \geq 1$ the rule for defining weights implies that

$$H(x) = H((x-1)/\alpha) + H((x-1)/\beta) \quad . \tag{4.5}$$

The basic relation (3.6) between H and M, namely

$$M(h(x)) = xH(x) - \int_0^x H(t)dt = xh(x) - \int_0^x h(t)dt \tag{4.6}$$

is still valid for this generalized H function. Indeed, by separating the strings σ which begin with L from those which begin with R (cf. (2.4)) we obtain the formula

$$M(h(x)) = h(x) + \alpha M(h((x-1)/\alpha)) + \beta M(h((x-1)/\beta)) \ . \tag{4.7}$$

Therefore if we can determine the asymptotic behavior of h (or H), we will be able to see how M grows, and to see how the value of k for which the minimum occurs in (0.1) depends on n.

Now that the problem has been set up in this way, it is comparatively easy to deduce the order of growth of M :

Lemma 4.1. Let γ be the positive constant defined by (4.3). There exist positive constants c_1, c_2, C_1, C_2, such that

$$c_1 x^\gamma \le H(x) \le c_2 x^\gamma \tag{4.8}$$

$$C_1 x^{1+1/\gamma} \le M(x) \le C_2 x^{1+1/\gamma} \tag{4.9}$$

for all sufficiently large x.

Proof: Choose c_2 so that $H(x) \le c_2 x^\gamma$ for $1 \le x < 2$. Then we can prove by induction on n that $H(x) \le c_2 x^\gamma$ for $1 \le x < n$, since $H(x) = H((x-1)/\alpha) + H((x-1)/\beta)$, which (by induction) is $\le c_2((x-1)/\alpha)^\gamma + c_2((x-1)/\beta)^\gamma = c_2(x-1)^\gamma < c_2 x^\gamma$.

The lower bound is a little trickier: If we assume that there is a positive constant a such that $H(x) \geq ax^{\gamma-\varepsilon}$ for $x < x_0$, then we have $H(x_0) \geq a(x_0-1)^{\gamma-\varepsilon} K$, where

$$K = \frac{\alpha^\varepsilon}{\alpha^\gamma} + \frac{\beta^\varepsilon}{\beta^\gamma} > \min(\alpha^\varepsilon, \beta^\varepsilon) > 1 \ . \tag{4.10}$$

For sufficiently large x_0 we will have $a(x_0-1)^{\gamma-\varepsilon} K \geq ax_0^{\gamma-\varepsilon}$ for $x_0 \leq x < x_0+1$. Indeed we can clearly extend this to all $x \geq x_0$. Since such an a exists for arbitrarily small ε, we must have $H(x)/x^{\gamma-\varepsilon} \to \infty$ as $x \to \infty$.

Let c be a constant such that $x^\gamma-(x-1)^\gamma \leq cx^{\gamma-1}$ for all large x; and let R be a constant such that $RK > R+c$, where $K = \alpha^{1-\gamma} + \beta^{1-\gamma} > 1$ as in (4.10). For sufficiently large x_0 we will have $(x_0-1)^{\gamma-1}RK \geq x_0^{\gamma-1}(R+c)$ and $H(x) > Rx^{\gamma-1}$ for all $x \geq x_0$. Thus there will be a positive constant $c_1 \leq 1$ such that

$$H(x) \geq c_1x^\gamma + Rx^{\gamma-1} \quad \text{for} \quad x_0 \leq x \leq \max(\alpha,\beta)x_0+1 \ . \tag{4.11}$$

We will show that this relation holds for all $x \geq x_0$. Let $x_n = \max(\alpha,\beta)x_0+n$; we will show by induction on n that (4.11) holds for $x_n \leq x \leq x_{n+1}$, and this will establish (4.8). The calculation is not difficult, and it reveals why we have been foresighted enough to choose c and R in such a mysterious way:

$$\begin{aligned} H(x) &= H((x-1)/\alpha) + H((x-1)/\beta) \\ &\geq c_1((x-1)^\gamma/\alpha^\gamma) + (x-1)^\gamma/\beta^\gamma)) + R((x-1)^{\gamma-1}/\alpha^{\gamma-1} + (x-1)^{\gamma-1}/\beta^{\gamma-1}) \\ &= c_1(x-1)^\gamma + RK(x-1)^{\gamma-1} \\ &\geq c_1x^\gamma - c_1\,c\,x^{\gamma-1} + (R+c)x^{\gamma-1} \\ &\geq c_1x^\gamma + Rx^{\gamma-1} \ . \end{aligned}$$

Now to obtain bounds on $M(x)$, we may use (2.7). By the definition of H we have

$$w(\sigma_n) \leq x \quad \text{if and only if} \quad H(x) > n \quad , \tag{4.12}$$

hence by (4.8)

$$c_2^{-1/\gamma}\, n^{1/\gamma} < w(\sigma_n) \leq c_1^{-1/\gamma}(n+1)^{1/\gamma} \tag{4.13}$$

for all large n. It follows that $M(n)$, the sum of the first n weights, satisfies

$$\liminf_{n\to\infty} \frac{M(n)}{n^{1+1/\gamma}} \geq \frac{\gamma}{\gamma+1} c_2^{-1/\gamma} \quad , \quad \limsup_{n\to\infty} \frac{M(n)}{n^{1+1/\gamma}} \leq \frac{\gamma}{\gamma+1} c_1^{-1/\gamma} . \tag{4.14}$$

The desired relation (4.9) is an immediate consequence. □

The latter part of this proof suggests the following result.

__Lemma 4.2.__ __Let__ γ __be as in Lemma 4.1. Then__ $\lim_{x\to\infty} H(x)/x^{\gamma}$ __exists if and only if__ $\lim_{n\to\infty} M(n)/n^{1+1/\gamma}$ __exists.__

__Proof:__ If $\lim_{x\to\infty} H(x)/x^{\gamma} = c$ then by (4.14), $\lim_{n\to\infty} M(n)/n^{1+1/\gamma} = (\gamma/(\gamma+1))c^{-1/\gamma}$. Conversely if $M(n) \sim Cn^{1+1/\gamma}$ we must have $w(\sigma_n) \sim (1+1/\gamma)Cn^{1/\gamma}$ since $w(\sigma_n)$ is a nondecreasing function of n. (This follows from a straightforward "Tauberian" argument: We have $M(\lfloor(1+\epsilon)n\rfloor) - M(n) \geq (\lfloor(1+\epsilon)n\rfloor - n)\sigma(w_n)$, hence $\limsup w(\sigma_n)/n^{1/\gamma} \leq C\epsilon^{-1}((1+\epsilon)^{1+1/\gamma} - 1)$ for all $\epsilon > 0$. Similarly, $\liminf w(\sigma_n)/n^{1/\gamma} \geq C\epsilon^{-1}(1-(1-\epsilon)^{1+1/\gamma})$.) Relation (4.12) completes the proof. □

Now let us investigate whether or not the limits do exist, for various α and β. We have seen that the limit does not exist when $\alpha = \beta$; similarly we can construct a large number of further examples, including all cases where α and β are integers and $\log \alpha / \log \beta$ is rational:

Theorem 4.1. *Let* $\min(\alpha,\beta) > 1$ *, and let* γ *be defined by (4.3). If* $\log\alpha/\log\beta$ *is rational, and if* $\gamma < 1$ *(i.e., if* $\alpha^{-1} + \beta^{-1} < 1$ *) then* $\lim_{n\to\infty} M(n)/n^{1+1/\gamma}$ *does not exist.*

Proof: We have $\alpha = \theta^p$, $\beta = \theta^q$ where p and q are relatively prime positive integers and $\theta > 1$. Without loss of generality we may assume that $p < q$. We will show that large "gaps" exist between weights, in the sense that there are positive real numbers $x < y$ such that no string weights lie between $\theta^m x$ and $\theta^m y-1$ for any m. This is enough to prove the theorem, since existence of the limit would imply that $H(\theta^n x)/H(\theta^n y-1) \to x^\gamma/y^\gamma \neq 1 = H(\theta^m x)/H(\theta^m y-1)$.

The weight of every string is a polynomial in θ, namely

$$\theta^{a_t} + \theta^{a_{t-1}} + \dots + \theta^{a_1} + \theta^{a_0} \quad , \tag{4.15}$$

where $a_0 = 0$ and $a_{i+1} - a_i = p$ or q for $0 \le i < t$. We may think of (4.15) as a number written with radix θ and digits 0 or 1, subject to the requirement that exactly $p-1$ or $q-1$ 0's occur between adjacent 1's. For convenience, we shall call (4.15) a *weight of order* a_t.

Let S be the set of all infinite expansions $\theta^{-b_0} + \theta^{-b_1} + \theta^{-b_2} + \dots$, where $b_0 = 0$ and $b_{i+1} - b_i = p$ or q for all $i \ge 0$. Thus S is a set of real numbers which satisfies

$$S = (1 + \theta^{-q}S) \cup (1 + \theta^{-p}S) \quad . \tag{4.16}$$

The largest element of S is $1/(1-\theta^{-p})$. This set contains large gaps, since the largest element of $1+\theta^{-q}S$ is $1+\theta^{-q}/(1-\theta^{-p})$ and this is smaller than the smallest element $1+\theta^{-p}/(1-\theta^{-q})$ of $1+\theta^{-p}S$. (We have $\theta^{-q}/(1-\theta^{-p}) < \theta^{-p}/(1-\theta^{-q})$ since this relation is equivalent to $\theta^{-q}-\theta^{-2q} < \theta^{-p}-\theta^{-2p}$, i.e., $(\theta^{-q}-\theta^{-p})(1-\theta^{-q}-\theta^{-p}) < 0$.) Equation (4.16) now shows that there are many further gaps.
$1+\theta^{-q}+\theta^{-q}S < 1+\theta^{-q}+\theta^{-p}S < 1+\theta^{-p}+\theta^{-q}S < 1+\theta^{-p}+\theta^{-p}S$,
etc., and we see that S is contained in something like a "Cantor ternary set": Every point not in S lies in an interval that is not in S , and S has measure zero.

Since every element of $\theta^{q-p}S$ is greater than every element of S , we can find positive numbers $x < y$ such that the interval (x,y) contains no points of

$$S_1 = \cdots \cup \theta^{-2}S \cup \theta^{-1}S \cup S \cup \theta S \cup \theta^2 S \cup \cdots \quad . \tag{4.17}$$

If w is the weight of a string such that $\theta^m x < w < \theta^m y-1$, then $w_1 = w+\theta^{-q}/(1-\theta^{-q}) \in S_1$, hence $\theta^{-m}w_1$ is an element of $S_1 \cap (x,y)$. This contradicts the choice of x and y , so there are no string weights between $\theta^m x$ and $\theta^m y-1$.

□

The next theorem shows why the hypothesis $\gamma < 1$ is necessary in Theorem 4.1, since there are infinitely many examples when $M(n)/n^{1+1/\gamma}$ approaches a limit even though $\log\alpha/\log\beta$ is rational.

__Theorem 4.2.__ __Let__ $\alpha^{-1}+\beta^{-1} = 1$. __If__ $\log\alpha/\log\beta$ __is rational and__ $\alpha \neq \beta$ __then__

$$M(n) \sim \frac{\beta-\alpha}{2(\log\beta-\log\alpha)}(\alpha^{-1}\log\alpha+\beta^{-1}\log\beta)n^2 \quad . \tag{4.18}$$

<u>Proof</u>: We have $\alpha = \theta^p$, $\beta = \theta^q$ where p and q are relatively prime positive integers and θ is the unique real root > 1 of the equation $1 - \theta^{-p} - \theta^{-q} = 0$. Since $\alpha \neq \beta$ we may assume that $p < q$. To prove this theorem we shall refine the observations made in the proof of Theorem 4.1 by studying the weights of order m more closely. Since p and q are relatively prime, there will be weights of order m for all large m.

The weights of order m have the form $\theta^m + w$, where w is a weight of order m-p or m-q. For large m, the largest weight of order m-q is

$$\theta^{m-q} + \theta^{m-q-p} + \theta^{m-q-2p} + \ldots + \theta^{uq} + \theta^{uq-q} + \ldots + 1$$

$$= \frac{\theta^{m-q+p} - \theta^{uq}}{\theta^p - 1} + \frac{\theta^{uq} - 1}{\theta^q - 1}$$

$$= \theta^m + a_u \quad ,$$

where

$$(u+1)q \equiv m \pmod{p} \quad , \quad 0 \leq u < p \quad , \text{ and}$$
$$a_u = \theta^{uq}(\theta^{p-q} - \theta^{q-p}) - \theta^{p-q} \quad . \tag{4.19}$$

Similarly the smallest weight of order m-p is $\theta^m + b_v$, where

$$(v+1)p \equiv m \pmod{q} \quad , \quad 0 \leq v < q \quad , \text{ and}$$
$$b_v = \theta^{vp}(\theta^{q-p} - \theta^{p-q}) - \theta^{q-p} \quad . \tag{4.20}$$

We have

$$a_u \leq a_0 = - \theta^{q-p} < - \theta^{p-q} = b_0 \leq b_v \quad , \tag{4.21}$$

hence the weights of order m appear in increasing order if we read their radix θ representations in lexicographic order.

Let $r = q-p$ and let $0 \leq s < r$. We shall divide the weights into r disjoint classes, where the weights of class s consist of all weights of order s , $s+r$, $s+2r$, ..., $s+kr$, ... From the argument in the preceding paragraph we see that the weights of class s appear in increasing order if we treat their radix θ representations as binary numbers; and furthermore the difference between consecutive weights of class s is bounded. (The set of all such differences contains pq elements $\{b_v - a_u \mid 0 \leq u < p , 0 \leq v < q\}$, plus perhaps a finite number of other differences which might appear for small m .)

Let f_m be the number of weights of order m , so that $f_0 = 1$, $f_m = 0$ for $m < 0$, and $f_m = f_{m-p} + f_{m-q}$ for $m > 0$. (In the special case $p = 1$, $q = 2$, f_m is a Fibonacci number and $\theta = (1+\sqrt{5})/2$.) Let $g_m = f_m + f_{m-r} + f_{m-2r} + \cdots$ be the number of weights of order $\leq m$ belonging to class $m \bmod r$; and finally let $h_0 = 1$ and $h_m = g_{m-r} - g_{m-r-q}$ for $m > 0$. We shall prove that if

$$w = \theta^{a_t} + \theta^{a_{t-1}} + \cdots + \theta^{a_1} + \theta^{a_0} \tag{4.22}$$

is the n-th smallest weight of class s , we have

$$n = h_{a_t} + h_{a_{t-1}} + \cdots + h_{a_1} + h_{a_0} \quad . \tag{4.23}$$

The proof is by induction on $m = a_t$; since $n = 1 = h_0$ when $m = 0$, we may assume that $m > 0$. Let (4.22) be the k-th smallest weight of order m . Then $n = g_m - f_m + k$, where $1 \leq k \leq f_m$. If $a_{t-1} = a_t - q$ then $w - \theta^m$ is the k-th smallest weight of order $m-q$, hence by induction (4.23) holds if and only if $n - h_m = g_{m-q} - f_{m-q} + k$. The latter is true by the definition of h_m and k . Similarly if $a_{t-1} = a_t - p$, $w - \theta^m$ is the $(k-f_{m-q})$-th smallest weight of order $m-p$, hence by induction (4.23) holds if and only if

$n - h_m = g_{m-p} - f_{m-p} + k - f_{m-q}$; since $g_{m-p} - f_{m-p} = g_{m-q}$, the proof of (4.23) is complete.

Note that the generating function for the h's is

$$\sum h_m z^m = 1 + \frac{(z^r - z^{r+q})}{(1-z^r)(1-z^p-z^q)} = \frac{(1-z^p)}{(1-z^r)(1-z^p-z^q)} \quad . \qquad (4.24)$$

This can be written

$$\sum h_m z^m = \frac{c}{1-\theta z} + R(z) \quad , \quad c = \frac{1-\theta^{-p}}{(1-\theta^{-r})(p\theta^{-p}+q\theta^{-q})} \quad , \qquad (4.25)$$

where $R(z)$ has no singularities in $|z| \le \theta^{-1}+\varepsilon$, since θ^{-1} is the smallest root of $1 - z^p - z^q = 0$. (If $1 = z^p + z^q$ then $1 \le |z|^p + |z|^q$, hence $|z| \ge \theta^{-1}$, with equality iff $z = \theta^{-1}$.) Consequently

$$h_m = c\theta^m + o(\theta^m) \quad . \qquad (4.26)$$

Let $H_s(x)$ be the number of weights of class s that are $\le x$, so that we have

$$H(x) = H_0(x) + H_1(x) + \ldots + H_{r-1}(x) + 1 \qquad (4.27)$$

For fixed s , we will show that $\lim_{x\to\infty} H_s(x)/x = c$. Let w be the largest weight of class s that is $\le x$; we have observed that x-w is bounded. If w is given by (4.22), $H_s(x)$ is given by (4.23), which equals $cw + o(w) = cx + o(x)$. It follows that

$$H(x) \sim r\,c\,x \quad , \qquad (4.28)$$

and the theorem is obtained by applying Lemma 4.2, since we have

$$rc = \frac{\log\beta - \log\alpha}{(\beta-\alpha)(\alpha^{-1}\log\alpha + \beta^{-1}\log\beta)} \quad .$$

□

A more detailed examination of the simplest case of Theorem 4.2, when $\alpha = \phi = (1+\sqrt{5})/2$ and $\beta = \phi^2$, actually yields an explicit formula for the n-th weight:

$$w(\sigma_n) = \phi^{-1} \lfloor n\phi^{-1} \rfloor + n \quad . \tag{4.29}$$

(Cf. [11], exercise 1.2.8-36, p. 493.) We also have

$$M(F_n) = \frac{1}{2\sqrt{5}} (\phi^{2n-1} - 2\phi^{n-1} - (-1)^n(2 - \phi^{-n+2})) + F_n \tag{4.30}$$

in this case, when $F_n = (\phi^n - (-\phi)^{-n})/\sqrt{5}$ is a Fibonacci number.

A completely different approach seems to be necessary when $\log \alpha / \log \beta$ is irrational. The following discussion is based on Dirichlet integrals.

<u>Theorem 4.3</u>. <u>Assume that</u> $\min(\alpha, \beta) > 1$ <u>and</u> $\log \alpha / \log \beta$ <u>is irrational, and let</u> γ <u>be defined by (4.3). Then</u> $\lim_{n \to \infty} M(n)/n^{1+1/\gamma}$ <u>exists</u>.

<u>Proof</u>: We shall make use of the following result from the analytic theory of numbers:

<u>Lemma 4.3</u>: <u>Let</u> $f(t)$ <u>be a nondecreasing function of the real variable</u> t, <u>with</u> $f(t) \geq 0$. <u>Assume that</u> $G(s) = \int_1^\infty f(t)dt / t^{s+1}$ <u>is an analytic function of the complex variable</u> s <u>when</u> $\text{Re}(s) \geq \gamma > 0$, <u>except for a first-order pole at</u> $s = \gamma$ <u>with positive residue</u> C. <u>Then</u> $f(t) \sim Ct^\gamma$.

A proof of this lemma appears in the appendix below. Let us apply this lemma to the function $f(t) = h(t) = H(t) - 1$. By Lemma 4.1, the integral

$$G(s) = \int_1^\infty h(t)dt / t^{s+1} \tag{4.31}$$

diverges when $s = \gamma$; but it converges absolutely and uniformly in any bounded region such that $\mathrm{Re}(s) \geq \gamma + \varepsilon$, for all fixed $\varepsilon > 0$. It follows that $G(s)$ is analytic in the half-plane $\mathrm{Re}(s) > \gamma$.

We will now show that $G(s)$ has a simple pole at $s = \gamma$, by analytically continuing G to the left of the line $\mathrm{Re}(s) = \gamma$. Consider the function $G_1(s) = (1 - \alpha^{-s} - \beta^{-s})G(s)$; when $\mathrm{Re}(s) > \gamma$, we have

$$\begin{aligned} G_1(s) &= \int_0^\infty h(t+1)\,dt \,/\, (t+1)^{s+1} - \int_\alpha^\infty h(t/\alpha)\,dt \,/\, t^{s+1} - \int_\beta^\infty h(t/\beta)\,dt \,/\, t^{s+1} \\ &= \int_0^\infty (1 + h(t/\alpha) + h(t/\beta))\,dt \,/\, (t+1)^{s+1} - \int_0^\infty (h(t/\alpha) + h(t/\beta))\,dt \,/\, t^{s+1} \\ &= \frac{1}{s} + \int_{\min(\alpha,\beta)}^\infty (h(t/\alpha) + h(t/\beta))\,dt\,((t+1)^{-s-1} - t^{-s-1}) \;. \end{aligned} \tag{4.32}$$

(This derivation uses (4.5) together with the fact that $h(t) = 0$ for $t < 1$.) Since $(t+1)^{-s-1} - t^{-s-1} = O((s+1)t^{-s-2})$, the latter integral converges whenever $\mathrm{Re}(s) > \gamma - 1$. Therefore we can analytically continue $G(s)$ into this region, by using the formula

$$G(s) = G_1(s)/(1 - \alpha^{-s} - \beta^{-s}) \quad , \tag{4.33}$$

and letting $G_1(s)$ be defined by (4.32). The only singularities of $G(s)$ in this region are the poles at $s = 0$ (if $\gamma < 1$) and possibly at the zeroes of $1 - \alpha^{-s} - \beta^{-s}$. For $s = \gamma$, we have a simple pole since we know this is a singularity of $G(s)$; the corresponding residue $G_1(\gamma)/(\ln \alpha \cdot \alpha^{-\gamma} + \ln \beta \cdot \beta^{-\gamma})$ must be positive, since $G(s)(s-\gamma)$ is positive when s approaches γ from the right. Furthermore this is the only singularity of $G(s)$ when $\mathrm{Re}(s) \geq \gamma$, for if we write $s = \sigma + i\tau$ we have $|\alpha^{-s} + \beta^{-s}| \leq \alpha^{-\sigma} + \beta^{-\sigma} \leq \alpha^{-\gamma} + \beta^{-\gamma} = 1$, where equality holds iff $\alpha^{-s} = \alpha^{-\gamma}$ and $\beta^{-s} = \beta^{-\gamma}$. This condition implies that $\tau = 2\pi p/\ln \alpha$ and $\tau = 2\pi q/\ln \beta$ for some integers p and q ; if τ is nonzero, this contradicts the fact that $\log \alpha/\log \beta$ is irrational.

We have now shown that $G(s)$ satisfies the hypotheses of Lemma 4.3, so $h(t) \sim Ct^{\sigma}$. This completes the proof (cf. Lemma 4.2). □

Incidentally if we attempt to apply this same method of proof when $\log \alpha/\log \beta$ is rational, we find that $1 - \alpha^{-s} - \beta^{-s}$ has infinitely many zeroes on the line $\mathrm{Re}(s) = \gamma$. But by an amazing coincidence, when $\gamma = 1$ and $\alpha \neq \beta$, $G_1(s)$ happens to be zero at all but one of these points.

It is possible to evaluate the residue C , when $\gamma = 1$; in fact, (4.18) holds also when $\log \alpha/\log \beta$ is irrational, since the residue is a continuous function of α .

The reader will note that Theorems 4.1 - 4.3 do not cover all cases. If $\gamma > 1$ and $\log \alpha/\log \beta$ is rational, we conjecture that $\lim M(n)/n^{1+1/\gamma}$ does not exist. It can be shown that this conjecture holds "almost always", with at most countably many counterexamples (see Fredman [6]).

5. The case $g(n) = 1$.

Another interesting case of the general problem we are considering occurs when $g(n) = 1$ for all n . The problem breaks into two subcases:

Theorem 5.1. *Assume that* $g(n) = 1$ *for all* n , *and that* $\min(\alpha,\beta) > 1$. *Let* γ *be defined by* (4.3). *Then there exist positive constants* C_1 , C_2 *such that* $C_1 n^{1+1/\gamma} < M(n) < C_2 n^{1+1/\gamma}$ *for all* n . *Furthermore* $\lim M(n)/n^{1+1/\gamma}$ *exists if and only if* $\log \alpha/\log \beta$ *is irrational.*

__Proof__: Theorem 1 applies to $M_{g\alpha\beta}$; in this case $D(n) = F(n)$ for $n \geq 1$, and it is easy to see that again we obtain a tree interpretation as in Section 2 above. This time we have $D(n) = (\alpha+\beta)w(\sigma_{n+1})$, where the weights are defined by the rules

$$w(\epsilon) = 1 \quad , \quad w(L\sigma) = \alpha w(\sigma) \quad , \quad w(R\sigma) = \beta w(\sigma) \quad . \tag{5.1}$$

The new rule is simpler than (2.4); the new weights are given by the final term of the previous weights, e.g. $w(LRRLL) = \alpha^3\beta^2$. The weight $\alpha^i\beta^j$ occurs $\binom{i+j}{i}$ times, since this is the number of strings containing i L's and j R's .

To deduce the asymptotic behavior of $M_{g\alpha\beta}(n)$, we proceed as above, letting $h(x)$ be the number of weights $\leq x$, and $H(x) = h(x)+1$. This time $H(x) = 1$ for $0 \leq x < 1$, and

$$H(x) = H(x/\alpha) + H(x/\beta) \quad , \quad \text{for } x \geq 1 \quad , \tag{5.2}$$

a relation simpler than (4.5). It is now easy to prove Lemma 4.1 for the new H and M functions, and Lemma 4.2 follows as before. Now we use the idea in the proof of Theorem 4.3: Let $G(s)$ be defined by (4.31), and $G_1(s) = (1-\alpha^{-s}-\beta^{-s})G(s)$. When $\mathrm{Re}(s) > \gamma$,

$$G_1(s) = \int_1^\infty (h(t) - h(t/\alpha) - h(t/\beta))\,dt\,/\,t^{s+1} = \int_1^\infty dt\,/\,t^{s+1} = 1/s \ . \tag{5.3}$$

Thus, $G(s)$ can be analytically extended to the entire plane by using the formula $G(s) = 1/\,s(1-\alpha^{-s}-\beta^{-s})$. When $\log\alpha/\log\beta$ is irrational, we argue as in Theorem 4.3 that Lemma 4.3 applies; hence $h(x) \sim Cx^\gamma$ and $M(n) \sim (\alpha+\beta)C^{-1/\gamma}\,\gamma n^{1+1/\gamma}\,/\,(\gamma+1)$, where $C = 1/(\alpha^{-\gamma}\ln\alpha^\gamma + \beta^{-\gamma}\ln\beta^\gamma)$. When $\log\alpha/\log\beta$ is rational, on the other hand, there is no analog to Theorem 4.2; the limit $h(x)/x^\gamma$ doesn't ever exist. The reason is that $\alpha = \theta^p$, $\beta = \theta^q$ for some θ , and all weights are powers of θ . Thus $h(\theta^n) = h(\theta^{n+1} - 0)$, i.e., there are large gaps between the weights, as in Theorem 4.1.

□

__Theorem 5.2.__ __If__ $g(n) = 1$ __for all__ n __and if__ $\alpha \le \beta$, $\alpha \le 1$, __then__

$$M_{g\alpha\beta}(n) = 1 + (\alpha+\beta)(1+\alpha+\ldots+\alpha^{n-1}) \quad . \tag{5.4}$$

__Proof__: For this case, Theorem 1 does __not__ apply, and in fact the function $M_{g\alpha\beta}$ turns out to be __concave__! We will prove (5.4) by induction. For $k < n/2$ we have $M(k) \le M(n-k)$, hence $\alpha M(k) + \beta M(n-k) \ge \alpha M(n-k) + \beta M(k)$. For $k > n/2$ we have $\alpha M(k) + \beta M(n-k) - (\alpha M(k-1) + \beta M(n-k+1)) = (\alpha+\beta)(\alpha^k - \beta\alpha^{n-k}) \le (\alpha+\beta)(\alpha^k - \alpha^{n+1-k}) \le 0$. Hence $M(n+1) = 1 + \alpha M(n) + \beta M(0)$, and the proof by induction is complete.

□

6. __The Case__ $g(n) = \delta_{n0}$.

When $g(0) = 1$ and $g(n) = 0$ for all $n > 0$, we obtain a case strongly related to the previous one. Let $M^*(n) = M_{g\alpha\beta}(n) - 1/(\alpha+\beta)$; then

$$M^*(0) = 1 - 1/(\alpha+\beta) \quad ,$$

$$M^*(n+1) = \min_{0 \le k \le n} (\alpha M^*(k) + \alpha/(\alpha+\beta) + \beta M^*(n-k) + \beta/(\alpha+\beta)) - 1/(\alpha+\beta)$$

$$= 1 - 1/(\alpha+\beta) + \min_{0 \le k \le n} (\alpha M^*(k) + \beta M^*(n-k)) \quad .$$

In other words $M^*(n) = M_{g^*\alpha\beta}(n)$, where $g^*(n) = 1 - 1/(\alpha+\beta)$. If $\alpha+\beta > 1$, $M^*(n)$ is therefore just $1 - 1/(\alpha+\beta)$ times the function in Theorem 5.1 or 5.2.

If $\alpha+\beta = 1$, $M(n)$ is trivially equal to 1 for all n. The remaining case has a new twist:

Theorem 6: *Let $g(n) = \delta_{n0}$ and $\alpha+\beta < 1$, and let γ be defined by (4.3).* (Note that γ is *negative*, between -1 and 0.) *If $\log \alpha/\log \beta$ is irrational,* $M_{g\alpha\beta}(n) \sim (\alpha+\beta-1)C^{-1/\gamma}\,\gamma n^{1+1/\gamma}/(\gamma+1)$, *where* $C = 1/(\alpha^{-\gamma}\ln \alpha^{\gamma} + \beta^{-\gamma}\ln \beta^{\gamma})$. *On the other hand if* $\log \alpha/\log \beta$ *is rational,* $\infty > \limsup M_{g\alpha\beta}(n)/n^{1+1/\gamma} > \liminf M_{g\alpha\beta}(n)/n^{1+1/\gamma} > 0$.

Proof: Theorem 1 applies to this case, since $D(0) = \alpha+\beta-1$ is *negative*. It follows that the D's are all negative (we have $M(n) > M(n+1)$, but M is still convex); in fact, $D(n) = (\alpha+\beta-1)/w(\sigma_{n+1})$, where the weights $w(\sigma)$ are defined now by the inverse rules

$$w(\epsilon) = 1\ ,\ w(L\sigma) = \alpha^{-1}w(\sigma)\ ,\ w(R\sigma) = \beta^{-1}w(\sigma)\ . \qquad (6.1)$$

The function $H(x)$ of Theorem 5.1 applies, but with α, β, γ replaced respectively by α^{-1}, β^{-1}, $-\gamma$; and we have $c_1x^{-\gamma} \le H(x) \le c_2x^{-\gamma}$. Therefore (cf. (4.13)) we have

$$(\alpha+\beta-1)c_2^{-1/\gamma}\,n^{1/\gamma} < D(n) \le (\alpha+\beta-1)c_1^{-1/\gamma}(n+1)^{1/\gamma}\ .$$

The theorem now follows as in Lemma 4.2 and Theorem 5.1, provided we can prove that $M(n) \to 0$ as $n \to \infty$, since $M(n) = 1 + D(0) + \dots + D(n-1)$. By definition, $M(n+1) \le \alpha M(k) + \beta M(n-k)$ for all k; and since the D's are negative, $M(n) < M(n-1)$. Hence

$$M(n+1) \le \alpha M(\lfloor n/2\rfloor) + \beta M(\lceil n/2\rceil)$$
$$\le \alpha M(\lfloor n/2\rfloor) + \beta M(\lfloor n/2\rfloor)\ .$$

But $\alpha+\beta < 1$, hence $M(n) \to 0$. □

7. The case $g(n) = n^2$.

We shall conclude this study of the M functions by considering a function $g(n)$ which grows more rapidly than those considered so far.

Theorem 7. *If* $g(n) = n^2$ *for all* n *and if* $\alpha^{-1} + \beta^{-1} > 1$ *then*

$$M_{g\alpha\beta}(n) \sim (\alpha+\beta)n^2 / (\alpha+\beta-\alpha\beta) \quad . \tag{7.1}$$

Proof: We may apply Theorem 1; and we note that $j+k = N-1$, whenever the "if" test occurs in the algorithm following that theorem. Therefore $F(n+1) = \min(\alpha D(k), \beta D(n-k))$ for some k . If $F(n+1) = \alpha D(k)$ then $\alpha D(j) \le F(n+1) \le \beta D(n-j)$ for all $j \le k$, and $\beta D(n-j) \le F(n+1) \le \alpha D(j)$ for all $j > k$; similarly if $F(n+1) = \beta D(n-k)$ we have $\alpha D(j) \le F(n+1) \le \beta D(n-j)$ for all $j < k$ and $\beta D(n-j) \le F(n+1) \le \alpha D(j)$ for all $j \ge k$. Thus in all cases

$$\min(\alpha D(j), \beta D(n-j)) \le F(n+1) \le \max(\alpha D(j), \beta D(n-j)) \tag{7.2}$$

for all j , $0 \le j \le n$. In particular, (7.2) holds when $j = \lfloor \beta n/(\alpha+\beta) \rfloor$. If we now write $D(n+1) = E(n) + 2Cn$, for $C = (\alpha+\beta)/(\alpha+\beta-\alpha\beta)$, we have $D(n+1) = g(n+2) - g(n+1) + F(n+1)$, hence

$$\begin{aligned}\min(\alpha D(\lfloor \beta n/(\alpha+\beta) \rfloor), \beta D(\lfloor \alpha n/(\alpha+\beta) \rfloor)) &\le E(n) + 2Cn - (2n+3) \\ &\le \max(\alpha D(\lfloor \beta n/(\alpha+\beta) \rfloor), \beta D(\lceil \alpha n/(\alpha+\beta) \rceil)) \quad .\end{aligned} \tag{7.3}$$

Now $2C\alpha \lfloor \beta n/(\alpha+\beta) \rfloor = 2Cn - 2n + O(1)$, so there is a constant A such that

$$|E(n)| \le \max(\alpha |E(\lfloor \beta n/(\alpha+\beta) \rfloor)|, \beta |E(\lceil \alpha n/(\alpha+\beta) \rceil)|) + A \quad . \tag{7.4}$$

From these relations we can prove that $|E(n)|$ does not grow too rapidly. Since $\alpha\beta/(\alpha+\beta) < 1$, there is a constant $\lambda < 1$ such that $\max(\alpha(\beta/(\alpha+\beta))^{\lambda}, \beta(\alpha/(\alpha+\beta))^{\lambda}) < 1$; let this maximum be ρ. There is a constant n_0 such that $\rho(n+\alpha+\beta)^{\lambda} + A \le n^{\lambda}$ for all $n \ge n_0$, and we can find $C_1 \ge 1$ with $|E(n)| \le C_1 n^{\lambda}$ for all $n < n_0$. By induction, (7.4) shows that $|E(n)| \le C_1 n^{\lambda}$ for all n.

We have proved that $D(n) = 2Cn + O(n^{\lambda})$; consequently $M(n) = Cn^2 + O(n^{\lambda+1})$.

□

When $\alpha+\beta = \alpha\beta$, i.e., $\alpha^{-1}+\beta^{-1} = 1$, we can use the same technique to show that $M_{g\alpha\beta}$ grows as $n^2 \log n$. Assume that $\alpha \le \beta$. If we write $D(n+1) = E_{\alpha}(n) + 2n \log n/\log \alpha$, we find

$$E_{\alpha}(n) \le \max(\alpha E_{\alpha}(\lfloor n/\alpha \rfloor), \beta E_{\alpha}(\lceil n/\beta \rceil)) + O(\log n) ;$$

this implies that $E_{\alpha}(n) \le C_{\alpha} n$ for some C_{α}. Similarly if we write $D(n+1) = E_{\beta}(n) + 2n \log n/\log \beta$, we find

$$E_{\beta}(n) \ge \min(\alpha E_{\beta}(\lfloor n/\alpha \rfloor), \beta E_{\beta}(\lceil n/\beta \rceil)) + O(\log n) ,$$

so $E_{\beta}(n) \ge -C_{\beta} n$. Therefore $M_{g\alpha\beta}(n)$ lies between $n^2 \log n/\log \beta + O(n^2)$ and $n^2 \log n/\log \alpha + O(n^2)$. It would be interesting to discover if $\lim M_{g\alpha\beta}(n)/n^2 \log n$ exists. In the case $\alpha = \beta = 2$ our derivation proves that

$$M_{g22}(n) = n^2 \log_2 n + O(n^2), \qquad (7.5)$$

a formula analogous to (3.1).

Incidentally when $\alpha = \beta = 2$ it is possible to give "explicit" formulas for $M(n)$, in terms of the binary representation of n. Let $n+1 = 2^{a_1} + 2^{a_2} + \dots + 2^{a_r}$, where $a_1 > a_2 > \dots > a_r \geq 0$. Then $D(n) = 1 + 2(a_1 \cdot 2^{a_1} + (a_2+1) \cdot 2^{a_2} + \dots + (a_r+1) \cdot 2^{a_r})$ and

$$M(n) = \sum_{1 \leq i,j \leq r} 2^{a_i+a_j}(\max(a_i,a_j) + 1 - 2\delta_{ij})$$

$$- 2^{a_1+1}(n-2^{a_1}) - \tfrac{2}{3}(2^{2a_1} - 1) + 2n - 1 . \qquad (7.6)$$

In particular, $M(2^a-1) = (a - \frac{5}{3}) \cdot 2^{2a} + 2^{a+1} - \frac{1}{3}$.

When $\alpha + \beta < \alpha\beta$ we have $(\alpha-1)(\beta-1) > 1$, so $\min(\alpha,\beta) > 1$. Now $g(n) = n^2 \geq n$, so we know from the results of Section 4 that $M_{g\alpha\beta}(n) \geq C_1 n^{1+1/\gamma}$ for some C_1, where $\alpha^{-\gamma} + \beta^{-\gamma} = 1$ (hence $\gamma < 1$, and $1 + 1/\gamma > 2$). It can be shown that $M_{g\alpha\beta}(n)$ is also $\leq C_2 n^{1+1/\gamma}$ in this case; in fact, whenever $\min(\alpha,\beta) > 1$, the general upper bound

$M_{g\alpha\beta}(n) = O(n^{1+1/\gamma})$ holds for <u>all</u> functions $g(n)$ that are $O(n^{1+1/\gamma-\varepsilon})$. This result will appear in a future paper [6].

Appendix. A Tauberian theorem

Now let us return to Lemma 4.3, on which we based our proofs of Theorems 4.3, 5.1 and 6. Results of this type were originally given by N. Wiener [17, 18] and S. Ikehara [9], in a rather complicated form somewhat more general than we need. Landau [12] simplified the ideas and used them to give a new proof of the prime number theorem; but he gave a slightly less general result than Lemma 4.3. The following proof is based on that of Landau, with minor modifications in order to prove what we need. (At this point, the reader should refer back to the statement of Lemma 4.3.)

Let $g(s) = G(s) - C/(s-\gamma)$, a function which is analytic for $\mathrm{Re}(s) \geq \gamma$. We now introduce two parameters, y and λ, which will eventually approach infinity. By the Riemann-Lebesgue lemma and the fact that g is analytic,

$$\left| \int_{-2}^{2} \left(1 - \frac{|t|}{2}\right) e^{i\lambda yt} \, g(\gamma + \varepsilon + i\lambda t)\,dt \right| \leq \frac{2K(\lambda)}{\lambda y}, \quad \text{for } 0 \leq \varepsilon \leq 1, \tag{A.1}$$

where $K(\lambda)$ depends only on λ. Let

$$\phi(x) = f(e^{y+x/\lambda})e^{-\gamma(y+x/\lambda)} \quad ; \tag{A.2}$$

then for fixed $0 < \varepsilon \leq 1$ and for $n \to \infty$ we have

$$\int_{-\lambda y}^{\lambda(n-y)} \phi(x) e^{-\varepsilon(y+x/\lambda)} \left(\frac{\sin x}{x}\right)^2 dx$$

$$= \frac{1}{2}\int_{-\lambda y}^{\lambda(n-y)} \phi(x) e^{-\varepsilon(y+x/\lambda)} \int_{-2}^{2} \left(1 - \frac{|t|}{2}\right) e^{-ixt}\, dt\, dx$$

$$= \frac{1}{2}\int_{-2}^{2}\left(1 - \frac{|t|}{2}\right) e^{i\lambda yt} \int_{-\lambda y}^{\lambda(n-y)} f(e^{y+x/\lambda}) e^{-(\gamma+\varepsilon+i\lambda t)(y+x/\lambda)}\, dx\, dt$$

$$= \frac{\lambda}{2}\int_{-2}^{2}\left(1 - \frac{|t|}{2}\right) e^{i\lambda yt} \int_{0}^{n} f(u) u^{-(\gamma+\varepsilon+i\lambda t+1)}\, du\, dt$$

$$= \frac{\lambda}{2}\int_{-2}^{2}\left(1 - \frac{|t|}{2}\right) e^{i\lambda yt}\, G(\gamma+\varepsilon+i\lambda t)\, dt + o(1) \quad ,$$

as $n \to \infty$. (The parameter n was introduced in order to justify the change in order of integration.) Note that in the special "ideal" case $f(u) = Cu^{\gamma}$ we have $\phi(x) = C$ and $G(s) = C/(s-\gamma)$; subtracting this particular case from the general case and combining the result with (A.1) yields

$$\left| \int_{-\lambda y}^{\infty} \phi(x) e^{-\varepsilon(y+x/\lambda)} \left(\frac{\sin x}{x}\right)^2 dx - C \int_{-\lambda y}^{\infty} e^{-\varepsilon(y+x/\lambda)} \left(\frac{\sin x}{x}\right)^2 dx \right| \leq \frac{K(\lambda)}{y} . \quad \text{(A.3)}$$

Now we can let $\varepsilon \to 0$, because if ε is extremely small the integrals clearly approach their value when $\varepsilon = 0$. Therefore we have proved

$$\left| \int_{-\lambda y}^{\infty} \phi(x) \left(\frac{\sin x}{x}\right)^2 dx - C \int_{-\lambda y}^{\infty} \left(\frac{\sin x}{x}\right)^2 dx \right| \leq \frac{K(\lambda)}{y} . \quad \text{(A.4)}$$

(This is the key inequality which gives us a handle on the problem: When y is very large and $|x|$ is bounded, $\phi(x)$ must be very nearly equal to C .)

From the monotonicity of f we now have the following inequalities when $-\sqrt{\lambda} \le x \le \sqrt{\lambda}$:

$$e^{-\gamma(y+1/\sqrt{\lambda})} f(e^{y-1/\sqrt{\lambda}}) \le \phi(x) \le f(e^{y+1/\sqrt{\lambda}})e^{-\gamma(y-1/\sqrt{\lambda})} \quad . \qquad (A.5)$$

Hence

$$\frac{f(e^{y-1/\sqrt{\lambda}})}{e^{\gamma(y-1/\sqrt{\lambda})}} \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} (\frac{\sin x}{x})^2 \, dx \le e^{\lambda\gamma/\sqrt{\lambda}} \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \phi(x) (\frac{\sin x}{x})^2 \, dx$$

$$\le e^{2\gamma/\sqrt{\lambda}} \left(C \int_{-\lambda y}^{\infty} (\frac{\sin x}{x})^2 \, dx + \frac{K(\lambda)}{y} \right) ,$$

for all fixed λ and $y > 1$. If we let $y \to \infty$ we obtain

$$\left(\limsup_{u \to \infty} \frac{f(u)}{u^\gamma} \right) \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} (\frac{\sin x}{x})^2 \, dx \le e^{2\gamma/\sqrt{\lambda}} \, C \int_{-\infty}^{\infty} (\frac{\sin x}{x})^2 \, dx \quad ,$$

and if we now let $\lambda \to \infty$ we have $\limsup f(u)/u^\gamma = C$. A similar argument, using the other half of (A.5), proves that $\liminf f(u)/u^\gamma = C$.

□

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