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NUMERICAL CALCULATION OF TRANSONIC FLOW PATTERNS

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1. Introduction.

In order to obtain flow patterns of incompressible fluids, one often uses the well-known fact that the real and imaginary parts of an analytic function $f(z)$, $z = x + iy$, are solutions of the Laplace equation. In particular $\operatorname{Im} f(z)$ is the stream function $\psi(x,y)$ of a possible flow pattern. If one wishes to investigate methods for determining the analytic function $f(z)$ which gives a flow in a prescribed channel or around a given profile, one is led to a boundary value problem which as a rule has a solution.

As a first step in the calculation of desired flow patterns it is instructive to compute the flow patterns which correspond to various known functions $f(z)$. These flow patterns may provide insight which suggests how to form combinations of stream functions which yield a desired flow. For example R. Legendre [11, 12] showed that various transcendental functions generate interesting flow patterns and he used these in the solution of certain flow problems.

In some methods for the numerical solution of boundary value problems, one first obtains a set of particular solutions, ψ_ν, of the differential equation and then one determines a set of coefficients A_ν so that

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$$\psi^{(N)} = \sum_{\nu=1}^{N} A_\nu \psi_\nu \tag{1.1}$$

approximates in some "best" sense the prescribed boundary values.

The aim of the present paper is to describe procedures for computing families of transonic flow patterns [illegible] compressible fluid. The partial differential equation for the stream function of such flows is nonlinear when considered in the physical plane but is linear if the equation is considered in the so-called hodograph plane (or in the H, θ plane which is related to the hodograph plane as described below). For this reason we shall determine a number of solutions of the differential equation for the stream function in the H, θ plane and we shall also calculate the corresponding flow patterns in the physical plane. It appears to be even more important to determine families of transonic flow patterns than it was to determine incompressible fluid flows because it can be shown that a flow pattern with boundary C does not necessarily exist for a given contour C. Moreover, while the determination of solutions of the Laplace equation is immediate, the determination of solutions of the compressibility equation in the H, θ plane (equation (1.6) or (1.7)) requires the use of methods which have been developed during the last twenty-five years. The numerical evaluation of these solutions poses problems which are only now being investigated.

We proceed now to a more detailed description of the method to be used in the present paper. As noted above, it is well-known that the partial differential equation satisfied by the stream function ψ of a two-dimensional compressible fluid flow when considered in the physical plane (i.e. when ψ is considered as a function of the coordinates x,y of the plane in which the flow takes place) is non-linear. If, however, we consider the equation in the hodograph plane, i.e. if we consider ψ as a function of the speed v and the angle θ which the velocity vector makes with a fixed direction (say with the x-axis), then as Chaplygin [9] and Molenbroek [14] have shown, ψ satisfies the linear partial differential equation

$$\frac{1-M^2}{\rho^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{v}{\rho}\frac{\partial}{\partial v}\left(\frac{v}{\rho}\frac{\partial\psi}{\partial v}\right) = 0 \tag{1.2}$$

Here ρ denotes the density which is related to the pressure p by the relation $p = c\rho^k$ where c and k are constants ($k = 1.4$ for air). The Mach number M and the density ρ are related to the velocity v according to the relations

$$\rho = \left(1 - \frac{k-1}{2}v^2\right)^{1/(k-1)} \tag{1.3}$$

$$M = v/\left(1 - \frac{k-1}{2}v^2\right)^{1/2} \tag{1.4}$$

The denominator in (1.4) being the local velocity of sound, M will be smaller or larger than one according as the flow is subsonic or supersonic. Instead of the velocity v we may introduce the variable

$$H = \int_{v_1}^{v} \frac{\rho}{v}\,dv = \int_{v_1}^{v}\left(1 - \frac{k-1}{2}v^2\right)^{1/(k-1)}\frac{dv}{v} \tag{1.5}$$

where v_1 denotes the velocity corresponding to $M = 1$. If one introduces the potential φ as well as the stream function ψ, then the differential equations connecting these quantities assume the form

$$(1.6) \qquad \frac{\partial \phi}{\partial \theta} = \frac{\partial \psi}{\partial H}, \quad \frac{\partial \phi}{\partial H} = -\ell(H)\frac{\partial \psi}{\partial \theta}, \quad \ell(H) = \frac{1-M^2}{\rho^2}$$

(See [1, 3, 4, 6, 13].)

From (1.6) we deduce at once that the differential equation for ψ may be written in the form

$$(1.7) \qquad \frac{\partial^2 \psi}{\partial H^2} + \ell(H)\frac{\partial^2 \psi}{\partial \theta^2} = 0$$

Equation (1.7) is a differential equation of mixed type since $\ell(H) > 0$ for $H < 0$ (i.e. for $v < v_1$, $M < 1$) while $\ell(H) < 0$ for $H > 0$ (i.e. for $v > v_1$, $M > 1$).

A representation of the solutions of equations (1.6) was obtained by Bergman [1, 2]. In a joint investigation Bers and Gelbart [7, 8] obtained independently of Bergman the same solutions. In their work the existence of these solutions follows as a consequence of their approach to more general partial differential equations by complex variable methods. Bers and Gelbart call the complex functions which they introduced by the name of Sigma-Monogenic functions. This method for obtaining a set of particular solutions of (1.6) and (1.7) is a generalization of the method of obtaining harmonic polynomials by taking real and imaginary parts of the analytic functions $(\theta + iH)^n$. These particular solutions $\varphi_{n,j}(H,\theta)$ and $\psi_{n,j}(H,\theta)$, $j = 1,2$, $n = 0,1,\ldots$ which satisfy (1.6) (and hence

(1.7) in the case of $\psi_{n,j}(H,\theta)$) may be written in the form

$$\varphi_{n,1}(H,\theta) + i\psi_{n,1}(H,\theta) = (\theta+iH)^{[n]}$$

(1.8)
$$\equiv [\theta^n - 2!\binom{n}{2}\theta^{n-2}\, p_2(H) + 4!\binom{n}{4}\theta^{n-4}\, p_4(H) + \cdots]$$
$$+ i[1!\binom{n}{1}\theta^{n-1}\, s_1(H) - 3!\binom{n}{3}\theta^{n-3}\, s_3(H) + \cdots]$$

$$\varphi_{n,2}(H,\theta) + i\psi_{n,2}(H,\theta) = i \odot (\theta+iH)^{[n]}$$

(1.9)
$$\equiv - [1!\binom{n}{1}\theta^{n-1}\, p_1(H) - 3!\binom{n}{3}\theta^{n-3}\, p_3(H) + \cdots]$$
$$+ i[\theta^n - 2!\binom{n}{2}\theta^{n-2}\, s_2(H) + 4!\binom{n}{4}\theta^{n-4}\, s_4(H) - \cdots]$$

where

(1.10)
$$p_0(H) = 1 \quad , \qquad p_1(H) = \int_{H_0}^{H} \ell(H_1)dH_1$$
$$p_m(H) = \int_{H_0}^{H} \ell(H_m) \int_{H_0}^{H_m} \int_{H_0}^{H_{m-1}} \ell(H_{m-2}) \int_{H_0}^{H_{m-2}} \cdots\, dH_1 dH_2 \cdots dH_m$$
$$m = 2,3,\ldots$$

(1.11)
$$s_0(H) = 1 \quad , \qquad s_1(H) = H - H_0 = \int_{H_0}^{H} dH_1$$
$$s_m(H) = \int_{H_0}^{H} \int_{H_0}^{H_m} \ell(H_{m-1}) \int_{H_0}^{H_{m-1}} \int_{H_0}^{H_{m-2}} \ell(H_{m-2}) \cdots dH_1 dH_2 \cdots dH_m$$
$$m = 2,3,\ldots$$

One easily deduces that $p_m(H)$ and $s_m(H)$ satisfy the recurrence relations

(1.10a) $$p_m(H) = \int_{H_0}^{H} \ell(H_m) \int_{H_0}^{H_m} p_{m-2}(H_{m-1}) dH_{m-1} dH_m$$

and

(1.11a) $$s_m(H) = \int_{H_0}^{H} \int_{H_0}^{H_m} \ell(H_{m-1}) \, s_{m-2}(H_{m-1}) dH_{m-1} dH_m$$

respectively. It will be convenient to choose $H_0 = 0$ corresponding to $M = 1$.

In Section 2 we approximate the function $\ell(H)$ by polynomials and calculate approximations $\tilde{\psi}_{n,j}$ to the stream functions $\psi_{n,j}$. In Section 3 we then form approximate stream functions which are linear combinations of these $\tilde{\psi}_{n,j}$ and we determine the flow patterns in the physical plane corresponding to these approximate stream functions.

The method of integral operators [5, Chap. 5] provides another method for generating particular solutions of the compressible flow equation. Some numerical calculations using this method have been carried out by Stark [18, 19].

Essentially the same partial differential equations arise in the study of magnetohydrodynamics. For the case of potential flow in a magnetic field perpendicular to the plane of the flow, and under some additional hypotheses, Nazarov [15, 16] has shown that the differential equations are essentially the same as the equation (1.2) for mixed compressible flow. He then showed how to determine various flow patterns using the method of integral operators.

2. Calculation of Stream Functions

In order to calculate the stream functions $\psi_{n,1}$ and $\psi_{n,2}$ given by (1.8) and (1.9) it is necessary to evaluate the functions $s_m(H)$ which are given by (1.11) in terms of multiple integrals involving the function $\ell(H)$. Because of the nature of the function $\ell(H)$ these integrals cannot be evaluated exactly. One possible method for the evaluation of these integrals is to approximate $\ell(H)$ in suitable intervals by polynomials in H. The intervals chosen were

$$(2.1)\qquad \begin{aligned} &J_1:\ \ -1.0 \le H \le 0 \quad \text{corresponding to} \quad .284 \le M \le 1 \\ &J_2:\ \ 0 \le H \le 0.2 \quad \text{corresponding to} \quad 1 \le M \le 1.78 \end{aligned}$$

The polynomial used to approximate $\ell(H)$ in J_κ is denoted by

$$(2.2)\qquad \mathcal{L}_\kappa(H) = \sum_{\nu=0}^{10} A_\nu^{(\kappa)} \eta_\kappa^\nu , \quad \eta_1 = H, \quad \eta_2 = 5H, \quad \kappa = 1,2 \quad .$$

In order to determine the coefficients of the approximating polynomial in these intervals we need an algorithm for the calculation of $\ell(H)$ as a function of H. We consider the special case of air for which $k = 1.4$. Since ρ and M are functions of v as shown by equations (1.3) and (1.4), then $\ell(H)$ can be expressed as a function of v using its definition in (1.6). In this special case the integral in (1.5) can be evaluated in closed form and yields

$$H = \left\{ (1-0.2v^2)^{1/2} \left[\frac{(1-0.2v^2)^2}{5} + \frac{1-0.2v^2}{3} + 1 \right] \right. \tag{2.3}$$

$$\left. - \frac{1}{2} \ln \left(\frac{1+(1-0.2v^2)^{1/2}}{1-(1-0.2v^2)^{1/2}} \right) \right\} \Bigg|_{v_1}^{v}$$

where $v_1 = \sqrt{5/6}$. From (2.3) we can devise an algorithm for the calculation of v corresponding to any value of H in J_1 or J_2 and then $\ell(H)$ can be calculated at once.

We then make use of the Remez algorithm [17] as adapted for the Burroughs B5500 computer by Golub and Smith [10] to determine the best tenth-degree polynomials which approximate $\ell(H)$ in the Chebyshev sense separately on the intervals J_1 and J_2. The values of the coefficients $A_\nu^{(\kappa)}$, $\kappa = 1,2$, $\nu = 0,1,\ldots,10$ are given in Table 1. Values of H, v, M, $\tilde{\ell}_\kappa(H)$, $\ell(H)$ and $\tilde{\ell}_\kappa(H) - \ell(H)$ are given in Table 2 for a representative set of values of H. From the approximation algorithm we know that

$$\max_{J_1} |\ell(H)-\tilde{\ell}_1(H)| \leq 1.7 \times 10^{-4} \tag{2.4}$$

$$\max_{J_2} |\ell(H)-\tilde{\ell}_2(H)| \leq 2.6 \times 10^{-3}$$

The values given in Table 2 confirm these bounds.

The function $\ell(H)$ becomes very large and negative when H exceeds 0.2. Indeed as $H \to 0.2513$, $M \to \infty$, $v \to \sqrt{5}$ and $\ell(H) \to -\infty$. It was therefore practically impossible to obtain a good polynomial approximation for $H > 0.2$.

Table 1. Values of the coefficients $A_\nu^{(\kappa)}$ of the approximations to $\ell(H)$.

ν \ κ	1	2
0	0.0001711222	-0.0025985374
1	-9.3681945493	-1.3189899758
2	-55.284916968	-22.598654078
3	-248.40627057	280.04321217
4	-840.40978809	-2002.2845890
5	-2052.0104328	8142.5278038
6	-3488.0712341	-20236.235647
7	-3982.8542755	31196.920595
8	-2896.2958023	-29181.568535
9	-1207.8174204	15188.597028
10	-219.39926509	-3388.9118171

TABLE 2. VALUES OF $\ell(H)$ AND ITS APPROXIMATING POLYNOMIALS

SUBSONIC

H	v	M	$\ell(H)$	$\tilde{\ell}_1(H)$	$\ell(H)-\tilde{\ell}_1(H)$
-1.0000	0.281675	0.283937	0.995929	0.995758	0.000171
-0.9500	0.296755	0.299404	0.994943	0.994945	-0.000002
-0.9000	0.312719	0.315823	0.993706	0.993550	0.000155
-0.8500	0.329633	0.333274	0.992148	0.992223	-0.000074
-0.8000	0.347570	0.351846	0.990181	0.990346	-0.000165
-0.7500	0.366611	0.371640	0.987685	0.987693	-0.000008
-0.7000	0.386848	0.392770	0.984504	0.984349	0.000155
-0.6500	0.408386	0.415372	0.980431	0.980299	0.000132
-0.6000	0.431344	0.439600	0.975181	0.975218	-0.000037
-0.5500	0.455859	0.465638	0.968369	0.968534	-0.000165
-0.5000	0.482091	0.493701	0.959460	0.959579	-0.000119
-0.4500	0.510227	0.524053	0.947697	0.947646	0.000051
-0.4000	0.540492	0.557009	0.931997	0.931828	0.000169
-0.3500	0.573154	0.592964	0.910772	0.910672	0.000100
-0.3000	0.608543	0.632414	0.881631	0.881721	-0.000089
-0.2500	0.647071	0.675994	0.840877	0.841045	-0.000167
-0.2000	0.689261	0.724542	0.782574	0.782577	-0.000003
-0.1500	0.735800	0.779194	0.696775	0.696605	0.000170
-0.1000	0.787612	0.841544	0.565895	0.565909	-0.000015
-0.0500	0.845996	0.913932	0.356635	0.356757	-0.000122
0.0000	0.912871	1.000000	0.000000	0.000171	-0.000171

SUPERSONIC

H	v	M	$\ell(H)$	$\tilde{\ell}_2(H)$	$\ell(H)-\tilde{\ell}_2(H)$
0.0000	0.912871	1.000000	0.000000	-0.002599	0.002599
0.0200	0.942622	1.039500	-0.214281	-0.216638	0.002357
0.0400	0.974465	1.082684	-0.493622	-0.491117	-0.002505
0.0600	1.008744	1.130295	-0.865912	-0.868276	0.002365
0.0800	1.045902	1.183327	-1.375578	-1.374987	-0.000591
0.1000	1.086529	1.243155	-2.096940	-2.095146	-0.001794
0.1200	1.131437	1.311754	-3.162109	-3.164673	0.002565
0.1400	1.181796	1.392111	-4.825058	-4.823672	-0.001385
0.1600	1.239389	1.489049	-7.627815	-7.627813	-0.000002
0.1800	1.307166	1.611131	-12.911118	-12.910861	-0.000257
0.2000	1.390636	1.775837	-24.834790	-24.832192	-0.002598

We may now calculate exactly the solutions $\tilde{\Psi}_{n,1}$ and $\tilde{\Psi}_{n,2}$ of the differential equation

$$(2.5) \qquad \tilde{\Psi}_{HH} + \mathcal{X}(H)\tilde{\Psi}_{\theta\theta} = 0 , \quad \mathcal{X}(H) = \mathcal{X}_\kappa(H) \text{ in } J_\kappa$$

which approximates the differential equation (1.7). If we use (1.11) and denote by $\tilde{s}_m(H)$ the polynomials which approximate $s_m(H)$ we find that

$$\tilde{s}_0(H) = 1 , \qquad \tilde{s}_1(H) = H$$

$$\tilde{s}_2(H) = \int_0^H \int_0^{H_2} \mathcal{X}(H_1)dH_1dH_2$$

$$\tilde{s}_3(H) = \int_0^H \int_0^{H_3} \mathcal{X}(H_2) \int_0^{H_2} dH_1dH_2dH_3$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(2.6) \qquad \tilde{s}_m(H) = \int_0^H \int_0^{H_m} \mathcal{X}(H_{m-1})\tilde{s}_{m-2}(H_{m-1})dH_{m-1}dH_m$$

If we make use of equation (2.2) and denote $\tilde{s}_m(H)$ by $\tilde{s}_m^{(\kappa)}(H)$ in J_κ we obtain

$$(2.7) \quad \tilde{s}_2^{(\kappa)}(H) = H^2 \sum_{\nu=0}^{10} c_\nu^{(2,\kappa)}\eta_\kappa^\nu , \quad c_\nu^{(2,\kappa)} = \frac{A_\nu^{(\kappa)}}{(\nu+1)(\nu+2)} , \quad \kappa = 1,2$$

$$(2.8) \quad \tilde{s}_3^{(\kappa)}(H) = H^3 \sum_{\nu=0}^{10} c_\nu^{(3,\kappa)}\eta_\kappa^\nu , \quad c_\nu^{(3,\kappa)} = \frac{A_\nu^{(\kappa)}}{(\nu+2)(\nu+3)} , \quad \kappa = 1,2$$

The general expression for $\tilde{s}_m(H)$ takes the form

$$(2.9) \qquad \tilde{s}_m^{(\kappa)}(H) = H^m \sum_{\nu=0}^{10[\frac{m}{2}]} C_\nu^{(m,\kappa)} \eta_\kappa^\nu \qquad \kappa = 1,2$$

where

$$(2.10) \qquad C_\nu^{(m,\kappa)} = \sum_{\{\xi | 0 \le \xi \le 10,\ 0 \le \nu-\xi \le 10[\frac{m-2}{2}]\}} \frac{A_\xi^{(\kappa)} C_{\nu-\xi}^{(m-2,\kappa)}}{(\nu+m-1)(\nu+m)}$$

Note that $[\frac{m}{2}] = \frac{m}{2}$ if m is even but $[\frac{m}{2}] = \frac{m-1}{2}$ if m is odd.

The approximate stream functions $\tilde{\Psi}_{n,1}$ and $\tilde{\Psi}_{n,2}$ can now be calculated using equations (1.8) and (1.9) with $s_m(H)$ replaced by $\tilde{s}_m(H)$.

It is seen from Table 1 that the coefficients of the polynomials $\mathcal{X}_\kappa(H)$ are very large; they are of constant sign in $\mathcal{X}_1(H)$ which is used for $H < 0$ and of alternating sign in $\mathcal{X}_2(H)$ which is used for $H > 0$. Hence in both cases the evaluation of these functions and of the functions $\tilde{s}_m^{(\kappa)}(H)$ will involve large terms of alternating sign and this may result in loss of several significant digits. The magnitude of the error given in (2.4) indicates that single precision arithmetic (about 11 decimal digits) on the Stanford Burroughs B5500 computer is adequate for the calculation of $\mathcal{X}_\kappa(H)$. However, for the evaluation of $\tilde{s}_m^{(\kappa)}(H)$ and the subsequent calculation of $\tilde{\Psi}_{n,j}$, single precision arithmetic was found to be inadequate. Hence these calculations were carried out in double precision arithmetic (78 bit arithmetic equivalent to about 22 decimal digits) on the B5500. Some typical values of $\tilde{\Psi}_{n,j}$ are given in Table 3. (Since $\tilde{\Psi}_{1,1} = H$, $\tilde{\Psi}_{2,1} = 2\theta H$, $\tilde{\Psi}_{1,2} = \theta$, these are not included in Table 3.)

It is important to investigate how well the approximate solutions $\tilde{\Psi}_{n,j}$ satisfy the differential equation (1.6). We therefore study the behavior of $S(\tilde{\Psi}_{n,j})$ where

$$(2.11) \qquad S(\tilde{\Psi}) = \frac{\partial^2\tilde{\Psi}}{\partial H^2} + \ell(H)\frac{\partial^2\tilde{\Psi}}{\partial\theta^2}$$

If we choose $\tilde{\Psi}$ to be a solution of (2.5) then it follows at once that

$$(2.12) \qquad S(\tilde{\Psi}) = [\ell(H)-\tilde{\ell}(H)]\frac{\partial^2\tilde{\Psi}}{\partial\theta^2} \; .$$

By formally differentiating equations (1.8) and (1.9) we can find formulas for $\partial^2\tilde{\Psi}_{n,j}/\partial\theta^2$ and these can be used to evaluate $S(\tilde{\Psi}_{n,j})$. These calculations were carried out in double precision arithmetic on the B5500. Typical values are given in Table 3.

It is seen that the values of $S(\tilde{\Psi}_{n,j})$ in Table 3 are small. It would be more meaningful to compare them with the magnitude of $\tilde{\Psi}_{n,j}$ and still better to consider the average values over some region G of the H,θ plane. We therefore introduce the quantity

$$(2.13) \qquad M_G(\tilde{\Psi}) = \left[\frac{\int_G\int|S(\tilde{\Psi})|^2 dH d\theta}{\int_G\int|\tilde{\Psi}|^2 dH d\theta}\right]^{1/2}$$

If one chooses

$$G = \{H,\theta|\ -1.0 \le H \le .2,\ \ 0 \le \theta \le 2\}$$

an extensive calculation shows that

$$M_G(\tilde{\psi}_{n,1}) \leq 0.01 , \qquad n \leq 7$$

$$M_G(\tilde{\psi}_{n,2}) \leq 0.01 , \qquad n \leq 5$$

TABLE 5. - VALUES OF $\tilde{\psi}_{n,j}$ AND $S(\tilde{\psi}_{n,j})$.

PSI(3, 1)

H	THETA 0.50 ψ	S(ψ)	THETA 1.00 ψ	S(ψ)	THETA 2.00 ψ	S(ψ)
-1.00	0.1633700	-0.0010265	-2.0866300	-0.0010265	-11.0866300	-0.0010265
-0.90	-0.0196152	-0.0008397	-2.0446152	-0.0008397	-10.1446152	-0.0008397
-0.80	-0.1489456	0.0007931	-1.9489456	0.0007931	-9.1489456	0.0007931
-0.70	-0.2307471	-0.0006517	-1.8057471	-0.0006517	-8.1057471	-0.0006517
-0.60	-0.2712086	0.0001343	-1.6212086	0.0001343	-7.0212086	0.0001343
-0.50	-0.2765697	0.0003584	-1.4015697	0.0003584	-5.9015697	0.0003584
-0.40	-0.2531522	-0.0004052	-1.1531522	-0.0004052	-4.7531522	-0.0004052
-0.30	-0.2073766	0.0001605	-0.8823766	0.0001605	-3.5823766	0.0001605
-0.20	-0.1457304	0.0000032	-0.5957304	0.0000032	-2.3957304	0.0000032
-0.10	-0.0746577	0.0000084	-0.2996577	0.0000084	-1.1996577	0.0000084
0.00	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.02	0.0150008	0.0002828	0.0600008	0.0002828	0.2400008	0.0002828
0.06	0.0450781	0.0008513	0.1800781	0.0008513	0.7200781	0.0008513
0.10	0.0757382	-0.0010744	0.3007382	-0.0010744	1.2007382	-0.0010744
0.14	0.1086399	-0.0011564	0.4236399	-0.0011564	1.6836399	-0.0011564
0.18	0.1488513	-0.0002686	0.5538513	-0.0002686	2.1738513	-0.0002686

PSI(4, 1)

H	THETA 0.50 ψ	S(ψ)	THETA 1.00 ψ	S(ψ)	THETA 2.00 ψ	S(ψ)
-1.00	1.3267400	-0.0020530	-0.3465201	-0.0041061	-24.5930402	-0.0082121
-0.90	0.8607697	-0.0016795	-0.9784606	-0.0033590	-23.5564213	-0.0067179
-0.80	0.5021087	0.0015863	-1.3957825	0.0031726	-21.9915650	0.0063451
-0.70	0.2385057	-0.0013034	-1.6229885	-0.0026067	-20.0459770	-0.0052135
-0.60	0.0575827	0.0002687	-1.6848346	0.0005373	-17.7496691	0.0010747
-0.50	-0.0531394	0.0007169	-1.6062788	0.0014338	-15.2125576	0.0028670
-0.40	-0.1063044	-0.0008105	-1.4126089	-0.0016210	-12.4252177	-0.0032420
-0.30	-0.1147532	0.0003211	-1.1295064	0.0006422	-9.4590129	0.0012844
-0.20	-0.0914608	0.0000063	-0.7829216	0.0000126	-6.3658431	0.0000253
-0.10	-0.0493153	0.0000168	-0.3986307	0.0000335	-3.1972613	0.0000671
0.00	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.02	0.0100016	0.0005656	0.0800033	0.0011312	0.6400065	0.0022624
0.06	0.0301561	0.0017026	0.2403123	0.0034052	1.9206246	0.0068105
0.10	0.0514764	-0.0021488	0.4029527	-0.0042976	3.2059054	-0.0085952
0.14	0.0772797	-0.0023129	0.5745595	-0.0046257	4.5091189	-0.0092515
0.18	0.1177026	-0.0005372	0.7754052	-0.0010745	5.8708104	-0.0021489

PSI(5, 1)

H	THETA 0.50 ψ	S(ψ)	THETA 1.00 ψ	S(ψ)	THETA 2.00 ψ	S(ψ)
-1.00	1.1593300	0.0005591	1.3221048	-0.0071308	-44.2767959	-0.0379352
-0.90	0.8937381	-0.0000610	1.5903748	-0.0063541	-46.2480003	-0.0315513
-0.80	0.6311412	0.0004922	0.2640490	0.0064408	-46.2043199	0.0302350
-0.70	0.3972384	-0.0007161	-0.6771151	-0.0056037	-44.3495289	-0.0251543
-0.60	0.2080588	0.0002024	-1.2635060	0.0012099	-40.8997653	0.0052400
-0.50	0.0711690	0.0006609	-1.5343538	0.0033492	-36.0814448	0.0141028
-0.40	-0.0131503	-0.0008549	-1.5367919	-0.0038943	-30.1313585	-0.0160516
-0.30	-0.0506833	0.0003699	-1.3247579	0.0015740	-23.2960561	0.0063904
-0.20	-0.0519131	0.0000077	-0.9573910	0.0000314	-15.0293028	0.0001261
-0.10	-0.0303953	0.0000209	-0.4965778	0.0000838	-7.9863077	0.0003354
0.00	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.02	0.0062520	0.0007070	0.1000081	0.0028280	1.6000325	0.0113120
0.06	0.0189453	0.0021320	0.3007808	0.0085168	4.8031231	0.0340560
0.10	0.0331006	-0.0027124	0.5073870	-0.0107705	8.0295124	-0.0430026
0.14	0.0529404	-0.0029913	0.7364894	-0.0116646	11.3456854	-0.0463575
0.18	0.0719096	-0.0007404	1.0395443	-0.0027550	14.9550833	-0.0108134

TABLE 3. - VALUES OF $\tilde{\psi}_{n,j}$ AND $S(\tilde{\psi}_{n,j})$

PSI[6, 1]

THETA	0.50		1.00		2.00	
H	ψ	S(ψ)	ψ	S(ψ)	ψ	S(ψ)
-1.00	-0.3388599	0.0068099	7.3978298	-0.0017780	-55.5999433	-0.1267378
-0.90	0.0792901	0.0040157	4.9268528	-0.0045647	-73.5001133	-0.1098984
-0.80	0.2381519	-0.0024890	2.7421193	0.0069191	-84.3892369	0.1090151
-0.70	0.2454509	0.0011102	0.9671948	-0.0075548	-88.7552657	-0.0933119
-0.60	0.1802196	-0.0000645	-0.3326905	0.0018861	-87.2104182	0.0198924
-0.50	0.0963554	0.0001905	-1.1433347	0.0057576	-80.4750336	0.0545288
-0.40	0.0263103	-0.0005385	-1.4946627	-0.0071557	-69.3675915	-0.0629409
-0.30	-0.0151670	0.0003071	-1.4534831	0.0030223	-54.7921591	0.0253101
-0.20	-0.0270873	0.0000072	-1.1151304	0.0000618	-37.7179078	0.0005026
-0.10	-0.0178975	0.0000207	-0.5931600	0.0001672	-19.1452399	0.0013407
0.00	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.02	0.0037520	0.0007071	0.1200103	0.0056562	3.8401302	0.0452483
0.06	0.0114455	0.0021393	0.3615621	0.0170483	11.5324931	0.1362535
0.10	0.0206111	-0.0027653	0.6147948	-0.0216467	19.3181711	-0.1722218
0.14	0.0356219	-0.0031917	0.9133418	-0.0237300	27.4634675	-0.1862320
0.18	0.0714722	-0.0008782	1.3632139	-0.0057857	36.7885837	-0.0438049

PSI[7, 1]

THETA	0.50		1.00		2.00	
H	ψ	S(ψ)	ψ	S(ψ)	ψ	S(ψ)
-1.00	-1.6603979	0.0083325	8.6363088	0.0238747	-3.9749361	-0.3181466
-0.90	-0.7746986	0.0058374	7.2288383	0.0103875	-74.7929822	-0.3020606
-0.80	-0.2621240	-0.0043801	5.1433063	-0.0018325	-126.3823142	0.3206572
-0.70	-0.0130929	0.0025887	2.9342525	-0.0044127	-158.8205474	-0.2890194
-0.60	0.0700845	-0.0003261	0.9924421	0.0019802	-172.9815309	0.0640997
-0.50	0.0661587	-0.0003571	-0.4432540	0.0076998	-170.4427244	0.1810654
-0.40	0.0316460	-0.0000933	-1.2704092	-0.0108986	-153.4073160	-0.2136858
-0.30	0.0005864	0.0001899	-1.5039546	0.0049626	-124.6141584	0.0872686
-0.20	-0.0129907	0.0000057	-1.2523910	0.0001058	-87.2163335	0.0017496
-0.10	-0.0101945	0.0000178	-0.6880418	0.0002915	-44.6083866	0.0046885
0.00	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.02	0.0021893	0.0006188	0.1400285	0.0098988	8.9604557	0.1583705
0.06	0.0067338	0.0018816	0.4227346	0.0298733	26.9237298	0.4770422
0.10	0.0125796	-0.0024894	0.7259455	-0.0381597	45.2138172	-0.6038872
0.14	0.0237533	-0.0030611	1.1093031	-0.0425850	64.7659495	-0.6560264
0.18	0.0554773	-0.0009602	1.7665290	-0.0108592	88.4834344	-0.1562217

PSI[8, 1]

THETA	0.50		1.00		2.00	
H	ψ	S(ψ)	ψ	S(ψ)	ψ	S(ψ)
-1.00	-1.2978519	-0.0032399	3.3942315	0.0708942	260.5541163	-0.5326284
-0.90	-0.8603520	0.0006907	6.1335915	0.0429059	50.3864492	-0.6400668
-0.80	-0.4549513	-0.0022037	6.1176754	-0.0253736	-119.2155195	0.7608800
-0.70	-0.1746287	0.0021328	4.5614798	0.0084041	-242.3325237	-0.7712069
-0.60	-0.0263269	-0.0003766	2.4492688	0.0006952	-316.8089715	0.1822394
-0.50	0.0243178	-0.0006447	0.4951412	0.0076500	-343.9154953	0.5384575
-0.40	0.0224257	0.0002488	-0.8669683	-0.0141331	-328.0002885	-0.6559222
-0.30	0.0053674	0.0000758	-1.4681934	0.0072598	-276.0623392	0.2736735
-0.20	-0.0056309	0.0000040	-1.3657653	0.0001643	-197.1878526	0.0055587
-0.10	-0.0056587	0.0000140	-0.7808918	0.0004643	-101.7870372	0.0149860
0.00	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.02	0.0012514	0.0004952	0.1600456	0.0158389	20.4814581	0.5067926
0.06	0.0038873	0.0015157	0.4843776	0.0478801	61.5799526	1.5271969
0.10	0.0075784	-0.0020668	0.8416296	-0.0616502	103.7251477	-1.9371814
0.14	0.0157638	-0.0027463	1.3289317	-0.0704146	149.9233276	-2.1173111
0.18	0.0430116	-0.0009957	2.2740323	-0.0189875	209.6047726	-0.5123954

TABLE 3. - VALUES OF $\tilde{\psi}_{n,j}$ AND $S(\tilde{\psi}_{n,j})$.

PSI[9, 1]

THETA	0.50		1.00		2.00	
H	ψ	S(ψ)	ψ	S(ψ)	ψ	S(ψ)
-1.00	0.5594215	-0.0204113	-9.5307435	0.1064768	1071.9889079	-0.0485403
-0.90	-0.1351562	-0.0086734	-0.0295244	0.0809407	561.7200719	-0.8374023
-0.80	-0.2530828	0.0031185	4.3397891	-0.0611904	103.7840640	1.5035859
-0.70	-0.1683341	-0.0001463	5.0482636	0.0327807	-265.2167729	-1.7743036
-0.60	-0.0644360	-0.0001883	3.6595486	-0.0026664	-522.7875642	0.4647486
-0.50	-0.0046504	-0.0005691	1.5419578	0.0038132	-659.9505955	1.4662673
-0.40	0.0111051	0.0003847	-0.3078639	-0.0154449	-680.2862579	-1.8650318
-0.30	0.0052635	-0.0000038	-1.3426341	0.0096581	-598.4482606	0.8002510
-0.20	-0.0020976	0.0000025	-1.4522642	0.0002373	-438.0219187	0.0165259
-0.10	-0.0030751	0.0000103	-0.8713846	0.0006924	-228.5618619	0.0448937
0.00	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.02	0.0007042	0.0003715	0.1800684	0.0237599	46.0843744	1.5204035
0.06	0.0022126	0.0011465	0.5465705	0.0719754	138.6509230	4.5840730
0.10	0.0045258	-0.0016219	0.9626637	-0.0935950	234.3789421	-5.8293474
0.14	0.0104338	-0.0023545	1.5772621	-0.1099573	342.3111736	-6.4197850
0.18	0.0333159	-0.0009941	2.9161851	-0.0316366	491.2786381	-1.5845289

PSI[10, 1]

THETA	0.50		1.00		2.00	
H	ψ	S(ψ)	ψ	S(ψ)	ψ	S(ψ)
-1.00	2.0252775	-0.0198234	-25.6910180	0.0527421	3035.6488573	4.0146562
-0.90	0.6780410	-0.0120396	-10.5880793	0.0858524	2027.1717658	0.7052990
-0.80	0.1032402	0.0067658	-0.7645980	-0.0909781	970.5230476	1.7729059
-0.70	-0.0563278	-0.0020386	3.7203833	0.0636997	16.5405824	-3.3840980
-0.60	-0.0511046	0.0000884	4.2031476	-0.0082255	-726.5200493	1.0639614
-0.50	-0.0154502	-0.0002615	2.5210694	-0.0053244	-1195.2328436	3.6982514
-0.40	0.0030272	0.0003408	0.3630294	-0.0131751	-1370.3551355	-4.9845324
-0.30	0.0036591	-0.0000431	-1.1285843	0.0117856	-1273.2501647	2.2160320
-0.20	-0.0005634	0.0000013	-1.5093863	0.0003235	-959.1198803	0.0467044
-0.10	-0.0016409	0.0000071	-0.9592010	0.0009824	-506.7507491	0.1280476
0.00	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.02	0.0003914	0.0002654	0.2000976	0.0339455	102.4124982	4.3440983
0.06	0.0012458	0.0008273	0.6093934	0.1030885	308.4000033	13.1058642
0.10	0.0026867	-0.0012213	1.0898966	-0.1356375	523.3804104	-16.7164031
0.14	0.0068949	-0.0019532	1.8599177	-0.1646594	773.4422103	-18.5760336
0.18	0.0257850	-0.0009640	3.7311824	-0.0509112	1142.7453078	-4.6919436

PSI[2, 2]

THETA	0.50		1.00		2.00	
H	ψ	S(ψ)	ψ	S(ψ)	ψ	S(ψ)
-1.00	-0.5216650	0.0003422	0.2283350	0.0003422	3.2283350	0.0003422
-0.90	-0.3588351	0.0003110	0.3911649	0.0003110	3.3911649	0.0003110
-0.80	-0.2158763	-0.0003305	0.5341237	-0.0003305	3.5341237	-0.0003305
-0.70	-0.0927193	0.0003103	0.6572807	0.0003103	3.6572807	0.0003103
-0.60	0.0107555	-0.0000746	0.7607555	-0.0000746	3.7607555	-0.0000746
-0.50	0.0947367	-0.0002390	0.8447367	-0.0002390	3.8447367	-0.0002390
-0.40	0.1595462	0.0003377	0.9095462	0.0003377	3.9095462	0.0003377
-0.30	0.2057550	-0.0001784	0.9557550	-0.0001784	3.9557550	-0.0001784
-0.20	0.2344084	-0.0000053	0.9844084	-0.0000053	3.9844084	-0.0000053
-0.10	0.2475960	-0.0000280	0.9975960	-0.0000280	3.9975960	-0.0000280
0.00	0.2500000	0.0051980	1.0000000	0.0051980	4.0000000	0.0051980
0.02	0.2500265	0.0047133	1.0000265	0.0047133	4.0000265	0.0047133
0.06	0.2508321	0.0047295	1.0008321	0.0047295	4.0008321	0.0047295
0.10	0.2545577	-0.0035813	1.0045577	-0.0035813	4.0045577	-0.0035813
0.14	0.2653657	-0.0027534	1.0153657	-0.0027534	4.0153657	-0.0027534
0.18	0.2928703	-0.0004974	1.0428703	-0.0004974	4.0428703	-0.0004974

TABLE 3. - VALUES OF $\tilde{\Psi}_{n,j}$ AND $S(\tilde{\Psi}_{n,j})$.

PSI(3, 2)

THETA	0.50		1.00		2.00	
M	Ψ	S(Ψ)	Ψ	S(Ψ)	Ψ	S(Ψ)
-1.00	-1.0324975	0.0005133	-1.3149951	0.0010265	3.3700099	0.0020530
-0.90	-0.7082527	0.0004465	-0.8265053	0.0009130	4.3469893	0.0018661
-0.80	-0.5738144	-0.0004957	-0.3976288	-0.0009914	5.2047423	-0.0019829
-0.70	-0.3890790	0.0004655	-0.0281579	0.0009310	5.9436842	0.0018620
-0.60	-0.2338667	-0.0001119	0.2822665	-0.0002239	6.5645331	-0.0004478
-0.50	-0.1078950	-0.0003584	0.5342100	-0.0007169	7.0684201	-0.0014338
-0.40	-0.0106807	0.0005066	0.7286385	0.0010131	7.4572771	0.0020262
-0.30	0.0588326	-0.0002676	0.8672651	-0.0005352	7.7345303	-0.0010703
-0.20	0.1016125	-0.0000079	0.9532251	-0.0000158	7.9064502	-0.0000316
-0.10	0.1213039	-0.0000419	0.9927879	-0.0000839	7.9855758	-0.0001677
0.00	0.1250000	0.0077970	1.0000000	0.0155940	8.0000000	0.0311880
0.02	0.1250398	0.0070700	1.0000796	0.0141399	8.0001592	0.0282799
0.06	0.1262482	0.0070942	1.0024964	0.0141885	8.0049927	0.0283769
0.10	0.1318366	-0.0053720	1.0136731	-0.0107440	8.0273463	-0.0214881
0.14	0.1480485	-0.0041301	1.0460971	-0.0082602	8.0921942	-0.0165205
0.18	0.1893054	-0.0007461	1.1286108	-0.0014923	8.2572217	-0.0029846

PSI(4, 2)

THETA	0.50		1.00		2.00	
M	Ψ	S(Ψ)	Ψ	S(Ψ)	Ψ	S(Ψ)
-1.00	-0.4745369	-0.0010710	-3.0095295	0.0004688	-1.8994999	0.0066279
-0.90	-0.4626686	-0.0006696	-2.2649266	0.0007299	1.7760413	0.0063282
-0.80	-0.4078118	0.0004281	-1.5667551	-0.0010591	5.0474718	-0.0070076
-0.70	-0.3270985	-0.0001726	-0.9318353	0.0012238	7.8992173	0.0068097
-0.60	-0.2352251	-0.0000048	-0.3743253	-0.0003407	10.3192739	-0.0016840
-0.50	-0.1443976	-0.0001358	0.0944175	-0.0012112	12.2996777	-0.0055125
-0.40	-0.0642527	0.0003233	0.4662051	0.0018430	13.8380364	0.0079216
-0.30	-0.0016996	-0.0002202	0.7366981	-0.0010229	14.9402889	-0.0042339
-0.20	0.0393869	-0.0000074	0.9067245	-0.0000311	15.6260751	-0.0001258
-0.10	0.0589006	-0.0000415	0.9855825	-0.0001673	15.9423098	-0.0006705
0.00	0.0625000	0.0077970	1.0000000	0.0311880	16.0000000	0.1247520
0.02	0.0625398	0.0070707	1.0001592	0.0282806	16.0006368	0.1131203
0.06	0.0637490	0.0071178	1.0049936	0.0284005	16.0199718	0.1135312
0.10	0.0693630	-0.0054700	1.0273727	-0.0215860	16.1094116	-0.0860502
0.14	0.0858594	-0.0043840	1.0925050	-0.0167743	16.3690876	-0.0663357
0.18	0.1293510	-0.0008741	1.2597673	-0.0031126	17.0314324	-0.0120663

PSI(5, 2)

THETA	0.50		1.00		2.00	
M	Ψ	S(Ψ)	Ψ	S(Ψ)	Ψ	S(Ψ)
-1.00	0.6178204	-0.0035329	-3.6143469	-0.0044996	-23.5285952	0.0115312
-0.90	0.2404163	-0.0024516	-3.1479308	-0.0025706	-12.8259684	0.0135198
-0.80	0.0201611	0.0018963	-2.5162499	0.0013141	-2.9850768	-0.0172004
-0.70	-0.0859480	-0.0012074	-1.8047907	-0.0000874	5.8272606	0.0184448
-0.60	-0.1149516	0.0001745	-1.0867369	-0.0002107	13.4718569	-0.0046991
-0.50	-0.0978357	0.0002578	-0.4226463	-0.0012766	19.8389081	-0.0168910
-0.40	-0.0594971	-0.0000361	0.1401021	0.0024606	24.8529750	0.0251836
-0.30	-0.0186366	-0.0001046	0.5683897	-0.0015471	28.4820821	-0.0137972
-0.20	0.0124463	-0.0000053	0.8454553	-0.0000502	30.7554124	-0.0004161
-0.10	0.0282617	-0.0000339	0.9759931	-0.0002775	31.8077440	-0.0022324
0.00	0.0312500	0.0064975	1.0000000	0.0519800	32.0000000	0.4158401
0.02	0.0312832	0.0058935	1.0002654	0.0471369	32.0021228	0.3770725
0.06	0.0322923	0.0059709	1.0083255	0.0474129	32.0665785	0.3785949
0.10	0.0370131	-0.0047215	1.0457091	-0.0363031	32.3648812	-0.2874870
0.14	0.0512342	-0.0040764	1.1552112	-0.0288034	33.2323641	-0.2228115
0.18	0.0912019	-0.0009418	1.4414309	-0.0056143	35.4550788	-0.0410744

TABLE 3. - VALUES OF $\tilde{\psi}_{n,j}$ AND $S(\hat{\psi}_{n,j})$.

PSI(6, 2)

THETA	0.50		1.00		2.00	
M	ψ	S(ψ)	ψ	S(ψ)	ψ	S(ψ)
-1.00	1.1133588	-0.0024360	-1.7736232	-0.0154471	-84.4775235	-0.0097500
-0.90	0.6492214	-0.0021585	-2.5622016	-0.0105664	-59.0863189	0.0082856
-0.80	0.3219300	0.0020216	-2.6744262	0.0077666	-34.2139736	-0.0250210
-0.70	0.1151490	-0.0015226	-2.3195610	-0.0043376	-10.8297827	0.0367700
-0.60	0.0046992	0.0002633	-1.6874585	0.0004190	10.2339032	-0.0115920
-0.50	-0.0368860	0.0005176	-0.9434308	-0.0003384	28.2922043	-0.0440877
-0.40	-0.0365965	-0.0003255	-0.2237876	0.0023616	42.8258656	0.0700976
-0.30	-0.0178341	0.0000345	0.3687346	-0.0019712	53.5111731	-0.0399767
-0.20	0.0020316	-0.0000031	0.7702353	-0.0000716	60.2744620	-0.0012337
-0.10	0.0133963	-0.0000247	0.9640398	-0.0004133	63.4234326	-0.0066851
0.00	0.0156250	0.0048731	1.0000000	0.0779700	64.0000000	1.2475203
0.02	0.0156499	0.0044215	1.0003980	0.0707109	64.0043685	1.1312401
0.06	0.0164083	0.0045225	1.0124947	0.0712965	64.1997611	1.1364931
0.10	0.0199970	-0.0037262	1.0687618	-0.0551907	65.0954355	-0.8654006
0.14	0.0312027	-0.0035461	1.2351548	-0.0451218	67.7064243	-0.6760627
0.18	0.0655245	-0.0009553	1.6814011	-0.0094004	74.4417677	-0.1270821

PSI(7, 2)

THETA	0.50		1.00		2.00	
M	ψ	S(ψ)	ψ	S(ψ)	ψ	S(ψ)
-1.00	0.4464706	0.0044380	2.9722533	-0.0259741	-223.9077192	-0.1690738
-0.90	0.4278564	0.0015702	0.0408533	-0.0205603	-175.9867600	-0.0837709
-0.80	0.3034076	-0.0001399	-1.5825563	0.0174627	-122.6816284	0.0207164
-0.70	0.1665806	-0.0005601	-2.1621379	-0.0117616	-68.0965376	0.0379754
-0.60	0.0626002	0.0001802	-1.9955837	0.0017032	-15.8758616	-0.0211136
-0.50	0.0041129	0.0004910	-1.3817143	0.0021209	30.8801310	-0.0995564
-0.40	-0.0156419	-0.0004219	-0.5933567	0.0009936	69.7022726	0.1762525
-0.30	-0.0121201	0.0000698	0.1459684	-0.0021292	98.8720646	-0.1066958
-0.20	-0.0012367	-0.0000014	0.6821422	-0.0000934	117.5991620	-0.0033994
-0.10	0.0062641	-0.0000166	0.9497492	-0.0005730	126.3863603	-0.0186732
0.00	0.0078125	0.0034112	1.0000000	0.1091580	128.0000000	3.4930568
0.02	0.0078299	0.0030964	1.0005573	0.0990058	128.0178320	3.1675563
0.06	0.0083623	0.0032072	1.0175046	0.1001461	128.5594269	3.1848260
0.10	0.0109195	-0.0027837	1.0966369	-0.0786460	131.0701774	-2.4341065
0.14	0.0192796	-0.0029624	1.3336058	-0.0667962	138.4128771	-1.9215505
0.18	0.0476520	-0.0009530	1.9905103	-0.0150590	157.5238794	-0.3703781

PSI(8, 2)

THETA	0.50		1.00		2.00	
M	ψ	S(ψ)	ψ	S(ψ)	ψ	S(ψ)
-1.00	-0.7431922	0.0106611	9.0849024	-0.0170070	-488.1153571	-0.8094140
-0.90	-0.1576505	0.0056536	4.2557871	-0.0223131	-428.3197002	-0.5145511
-0.80	0.0601951	-0.0029789	0.8208215	0.0247473	-335.6181664	0.3165924
-0.70	0.0938495	0.0010005	-1.1527736	-0.0201550	-223.8873185	-0.0941015
-0.60	0.0595074	-0.0000098	-1.8603734	0.0035262	-106.0258446	-0.0213853
-0.50	0.0193905	0.0002468	-1.6511819	0.0063125	6.3884812	-0.1893028
-0.40	-0.0028237	-0.0003460	-0.9328852	-0.0021161	103.8053357	0.4049504
-0.30	-0.0067235	0.0000891	-0.0901534	-0.0018417	179.1288441	-0.2672753
-0.20	-0.0017496	-0.0000003	0.5824978	-0.0001135	228.3664747	-0.0088829
-0.10	0.0028836	-0.0000105	0.9331547	-0.0007546	251.6994570	-0.0496449
0.00	0.0039063	0.0022741	1.0000000	0.1455440	256.0000000	9.3148181
0.02	0.0039179	0.0020654	1.0007431	0.1320253	256.0475523	8.4470969
0.06	0.0042741	0.0021729	1.0233594	0.1340802	257.4921248	8.5016895
0.10	0.0060169	-0.0020053	1.1294681	-0.1071730	264.1970055	-6.5276195
0.14	0.0120356	-0.0024056	1.4521856	-0.0952249	283.8842189	-5.2198640
0.18	0.0349473	-0.0009131	2.3831222	-0.0234227	335.6928553	-1.0368814

TABLE 3. - VALUES OF $\tilde{\Psi}_{n,j}$ AND $S(\tilde{\Psi}_{n,j})$.

PSI(9, 2)

THETA	0.50		1.00		2.00	
M	Ψ	S(Ψ)	Ψ	S(Ψ)	Ψ	S(Ψ)
-1.00	-1.2133478	0.0054818	12.6652964	0.0365521	-874.4017672	-2.7584680
-0.90	-0.5403117	0.0047903	8.3733405	0.0004562	-894.4433360	-1.9704558
-0.80	-0.1672623	-0.0036097	3.9732432	0.0188278	-788.8941915	1.4595569
-0.70	-0.0099840	0.0018610	0.6386254	-0.0241549	-595.9344597	-0.7607576
-0.60	0.0271185	-0.0001682	-1.2081908	0.0053615	-354.5353439	0.0426525
-0.50	0.0174598	-0.0000354	-1.6800279	0.0118865	-101.6030539	-0.2656524
-0.40	0.0026182	-0.0001902	-1.2062774	-0.0072137	130.5827630	0.8473974
-0.30	-0.0030737	0.0000778	-0.3287690	-0.0009374	316.7869395	-0.6349448
-0.20	-0.0014036	0.0000002	0.4728493	-0.0001293	441.2564767	-0.0222829
-0.10	0.0013031	-0.0000063	0.9142960	-0.0009558	500.9491602	-0.1271948
0.00	0.0019531	0.0014619	1.0000000	0.1871280	512.0000000	23.9523893
0.02	0.0019606	0.0013286	1.0009554	0.1697738	512.1222784	21.7219709
0.06	0.0021905	0.0014238	1.0300644	0.1732418	515.8378791	21.8887130
0.10	0.0033406	-0.0014078	1.1674169	-0.1413877	533.1084816	-16.8986748
0.14	0.0075695	-0.0019110	1.5928936	-0.1321908	584.0629343	-13.7198787
0.18	0.0257881	-0.0008540	2.8778246	-0.0356539	719.9196076	-2.8210583

PSI(10, 2)

THETA	0.50		1.00		2.00	
M	Ψ	S(Ψ)	Ψ	S(Ψ)	Ψ	S(Ψ)
-1.00	-0.4087223	-0.0115214	8.7328581	0.1396283	-1150.9249996	-7.5178146
-0.90	-0.3939215	-0.0022069	9.6536534	0.0595592	-1587.7538305	-5.9946804
-0.80	-0.2108838	-0.0008952	6.6619251	-0.0122068	-1643.9983741	4.9911078
-0.70	-0.0679881	0.0013106	2.8274878	-0.0160981	-1402.5194118	-3.1262375
-0.60	-0.0034079	-0.0001998	-0.0813693	0.0062478	-963.0764736	0.3560739
-0.50	0.0091792	-0.0002085	-1.4258316	0.0177556	-430.0154189	-0.0686977
-0.40	0.0035683	-0.0000429	-1.3806578	-0.0141768	98.9535303	1.5775018
-0.30	-0.0010553	0.0000580	-0.5584002	0.0007237	543.0690865	-1.4379189
-0.20	-0.0009282	0.0000004	0.3549450	-0.0001380	848.0545812	-0.0540891
-0.10	0.0005757	-0.0000036	0.8932192	-0.0011739	996.3953462	-0.3166369
0.00	0.0009766	0.0009137	1.0000000	0.2339101	1024.0000000	59.8809732
0.02	0.0009812	0.0008310	1.0011943	0.2122567	1024.3056990	54.3074487
0.06	0.0011257	0.0009096	1.0376256	0.2177983	1033.5975764	54.8012246
0.10	0.0018665	-0.0009697	1.2106738	-0.1820257	1076.8601577	-42.5781384
0.14	0.0047868	-0.0014913	1.7581425	-0.1799310	1205.2130064	-35.1742668
0.18	0.0191189	-0.0007832	3.4983495	-0.0533627	1552.6267527	-7.5149879

3. Examples of Families of Flow Patterns in the H,θ and Physical Planes.

In this section we construct a number of examples of particular solutions of equation (2.5) in the H,θ plane and we show how to obtain the corresponding flows in the physical plane.

In order to transform a solution from the H,θ plane to the physical plane we make use of the relations

$$x = \int \left[\frac{(M^2-1)\cos\theta}{\rho v^2}\psi_\theta - \frac{\sin\theta}{\rho v}\psi_v\right]dv + \left[\frac{\cos\theta}{\rho}\psi_v - \frac{\sin\theta}{\rho v}\psi_\theta\right]d\theta$$

(3.1)

$$y = \int \left[\frac{(M^2-1)\sin\theta}{\rho v^2}\psi_\theta + \frac{\cos\theta}{\rho v}\psi_v\right]dv + \left[\frac{\sin\theta}{\rho}\psi_v + \frac{\cos\theta}{\rho v}\psi_\theta\right]d\theta$$

We observe that $\psi_v dv + \psi_\theta d\theta = 0$ along a streamline. Hence when computing a streamline $\psi(x,y)$ = constant, if we integrate along the streamline, the transformation equations (3.1) simplify to

$$x = \int \frac{\cos\theta}{\rho}\left[\frac{M^2-1}{v^2}\psi_\theta dv + \psi_v d\theta\right]$$

(3.2)

$$y = \int \frac{\sin\theta}{\rho}\left[\frac{M^2-1}{v^2}\psi_\theta dv + \psi_v d\theta\right] .$$

We note that if the Jacobian of this transformation vanishes at any point the transformation fails to be one-to-one in a neighborhood of such a point and the flow may have no physical significance in the neighborhood. See e.g. [6, p. 17] or [13, p. 311ff.].

The following five particular solutions of the equation (2.5) were investigated:

$$\tilde{\Psi}^{(1)} = 5\,\tilde{\Psi}_{1,1} + \tilde{\Psi}_{5,1} + \tilde{\Psi}_{4,2}$$

$$\tilde{\Psi}^{(2)} = 5\,\tilde{\Psi}_{1,1} + \tilde{\Psi}_{4,1} + \tilde{\Psi}_{3,2}$$

$$\tilde{\Psi}^{(3)} = 5\,\tilde{\Psi}_{1,1} + \tilde{\Psi}_{4,1} + \tilde{\Psi}_{4,2}$$

$$\tilde{\Psi}^{(4)} = 5\,\tilde{\Psi}_{1,1} + \tilde{\Psi}_{2,1} + \tilde{\Psi}_{3,1} + \tilde{\Psi}_{4,1} + \tilde{\Psi}_{4,2}$$

$$\tilde{\Psi}^{(5)} = 5\,\tilde{\Psi}_{1,1} + \tilde{\Psi}_{3,1} + \tilde{\Psi}_{3,2}$$

Table 4 gives sets of points in the H,θ plane at which these solutions assume certain chosen values. The corresponding values of x,y in the physical plane are also given in this table. The streamlines are drawn in both the hodograph or v, θ-plane and in the physical plane in Figures 1-5.

As noted above the transformation from the H, θ-plane to the physical plane may fail to be one-to-one. Consequently the flows represented by the above streamfunctions correspond to physically possible flows only in certain subdomains. This explains why some of the streamlines drawn intersect. It should also be observed that not all of the intersections of the streamlines with the sonic line (shown as a dotted line) correspond to transitions between subsonic and supersonic flow. Those intersections which do correspond to such transitions are indicated on the figures.

TABLE V. - VALUES OF COORDINATES ALONG CERTAIN STREAMLINES.

$$\tilde{\psi}^{(1)} = 5\tilde{\psi}_{1,1} + \tilde{\psi}_{5,1} + \tilde{\psi}_{4,2}$$

$\tilde{\psi}^{(1)} = 1.00000$

H	θ	V	X	Y
-0.175076	2.010000	0.711066	-1.052211	53.880153
-0.169031	1.890000	0.717521	3.305738	41.527489
-0.161098	1.770000	0.725052	6.912848	31.425043
-0.150596	1.650000	0.735216	7.160847	23.392230
-0.136604	1.530000	0.749118	7.298107	17.180735
-0.117917	1.410000	0.768367	6.839968	12.509417
-0.093065	1.290000	0.795281	6.084777	9.094026
-0.060529	1.170000	0.833071	5.234820	6.671440
-0.019442	1.050000	0.885698	4.420842	5.014644
0.000000	1.000000	0.912871	4.113093	4.509968
0.020428	0.950000	0.943280	3.829896	4.090624
0.045562	0.890000	0.983737	3.525699	3.689958
0.070335	0.830000	1.027553	3.259283	3.370730
0.093569	0.770000	1.073043	3.024370	3.137340
0.114319	0.710000	1.118186	2.809629	2.940971
0.132100	0.650000	1.161157	2.601914	2.772872
0.146694	0.590000	1.200715	2.390294	2.621764
0.158082	0.530000	1.236245	2.168380	2.482649
0.168761	0.470000	1.267586	1.934475	2.354873
0.176627	0.410000	1.294834	1.690329	2.230931
0.182918	0.350000	1.318182	1.439623	2.130772
0.187903	0.290000	1.337820	1.186467	2.055838
0.191779	0.230000	1.353877	0.934337	1.988716
0.194877	0.170000	1.366386	0.685437	1.938212
0.196674	0.110000	1.375282	0.440480	1.903851
0.197809	0.050000	1.380443	0.198829	1.884252
0.198092	-0.010000	1.381743	-0.041122	1.870445
0.197525	-0.070000	1.379148	-0.281424	1.889076
0.196105	-0.130000	1.372729	-0.524036	1.913450
0.193814	-0.190000	1.362613	-0.770297	1.953236
0.190601	-0.250000	1.348919	-1.020426	2.009220
0.186373	-0.310000	1.331676	-1.273214	2.081958
0.180978	-0.370000	1.310820	-1.525988	2.171410
0.174195	-0.430000	1.286195	-1.774960	2.276694
0.165732	-0.490000	1.257602	-2.016052	2.396136
0.155230	-0.550000	1.224865	-2.246018	2.527792
0.142281	-0.610000	1.187938	-2.464092	2.670666
0.126512	-0.670000	1.147166	-2.673455	2.826602
0.107719	-0.730000	1.103300	-2.882117	3.002530
0.086062	-0.790000	1.057817	-3.102122	3.211995
0.062192	-0.850000	1.012667	-3.346480	3.474474
0.037166	-0.910000	0.969813	-3.625105	3.812378
0.012165	-0.970000	0.930734	-3.942423	4.247998
-0.011807	-1.030000	0.896178	-4.297322	4.802162
-0.034073	-1.090000	0.866261	-4.684621	5.495080
-0.054277	-1.150000	0.840701	-5.096266	6.347819
-0.072308	-1.210000	0.819030	-5.521732	7.383514
-0.088214	-1.270000	0.800724	-5.947746	8.627994
-0.102132	-1.330000	0.785279	-6.357635	10.109910
-0.114246	-1.390000	0.772246	-6.730530	11.860514
-0.124755	-1.450000	0.761231	-7.040549	13.913231
-0.133853	-1.510000	0.751902	-7.256019	16.303073
-0.141724	-1.570000	0.743982	-7.338782	19.065957
-0.148535	-1.630000	0.737237	-7.243450	22.237924
-0.154433	-1.690000	0.731476	-6.917029	25.854279
-0.159548	-1.750000	0.726538	-6.298235	29.948649
-0.163990	-1.810000	0.722292	-5.317184	34.551957
-0.167856	-1.870000	0.718629	-3.895059	39.691319
-0.171228	-1.930000	0.715457	-1.943904	45.388858
-0.174177	-1.990000	0.712702	0.033472	51.660444

-0.129059	2.010000	0.756793	-0.179872	50.448310
-0.111143	1.890000	0.775550	-3.315297	38.841535
-0.087433	1.770000	0.801606	-1.910747	29.507982
-0.055705	1.650000	0.838948	0.926884	22.260376
-0.012878	1.530000	0.894693	0.148902	16.923637
0.000000	1.500000	0.912871	0.090891	15.873015
0.023686	1.450000	0.948293	-0.051279	14.377968
0.055810	1.390000	1.001341	-0.251090	13.038430
0.091502	1.330000	1.068797	-0.408804	12.285894
0.128525	1.270000	1.152151	-0.404195	12.283764
0.162699	1.210000	1.247856	-0.082243	13.282427
0.190236	1.150000	1.347397	0.704200	15.093874

$\tilde{\psi}^{(1)} = 0.00250$

H	θ	V	X	Y
-0.185674	2.010000	0.702116	-0.104186	54.684770
-0.182345	1.890000	0.705156	4.926752	42.109018
-0.177999	1.770000	0.709156	7.983521	31.810439
-0.172278	1.650000	0.714475	8.957835	23.599095
-0.164703	1.530000	0.721615	8.998680	17.202913
-0.154659	1.410000	0.731257	8.322380	12.351674
-0.141423	1.290000	0.744283	7.727554	8.760058
-0.124292	1.170000	0.761710	6.813909	6.159442
-0.102911	1.050000	0.784430	5.902371	4.309127
-0.077859	0.930000	0.812559	5.055445	3.009340
-0.051271	0.810000	0.844416	4.493977	2.101805
-0.026743	0.690000	0.875896	3.009187	1.461050
-0.007749	0.570000	0.901790	2.974356	0.964727
0.000000	0.500000	0.912871	2.915099	0.783639
0.004129	0.450000	0.918856	2.360016	0.652411
0.007695	0.390000	0.924038	2.053541	0.515496
0.010009	0.330000	0.927517	1.745108	0.399403
0.011376	0.270000	0.929554	1.434048	0.303135
0.012087	0.210000	0.930618	1.119717	0.226204
0.012392	0.150000	0.931075	0.802333	0.169446
0.012486	0.090000	0.931215	0.482445	0.129873
0.012500	0.030000	0.931236	0.160919	0.110558
0.012499	-0.030000	0.931234	-0.161174	0.110558
0.012486	-0.090000	0.931215	-0.482700	0.129873
0.012392	-0.150000	0.931075	-0.802598	0.169446
0.012087	-0.210000	0.930618	-1.119972	0.226204
0.011376	-0.270000	0.929554	-1.434303	0.303135
0.010009	-0.330000	0.927517	-1.745453	0.399403
0.007695	-0.390000	0.924088	-2.053796	0.515496
0.004129	-0.450000	0.918850	-2.360271	0.652411
-0.000965	-0.510000	0.911443	-2.666455	0.811893
-0.007789	-0.570000	0.901790	-2.974616	0.994734
-0.016414	-0.630000	0.889825	-3.407744	1.211131
-0.026743	-0.690000	0.875896	-3.009448	1.461057
-0.038501	-0.750000	0.860537	-3.943675	1.754544
-0.051271	-0.810000	0.844416	-4.204238	2.101812
-0.064555	-0.870000	0.828224	-4.664263	2.515249
-0.077859	-0.930000	0.812559	-5.055705	3.010347
-0.090752	-0.990000	0.797869	-5.468974	3.600705
-0.102911	-1.050000	0.784430	-5.902632	4.309134
-0.114126	-1.110000	0.772373	-6.353091	5.152818
-0.124292	-1.170000	0.761710	-6.814170	6.159450
-0.133380	-1.230000	0.752382	-7.276695	7.351282
-0.141423	-1.290000	0.744283	-7.727825	8.760066
-0.148487	-1.350000	0.737285	-8.150469	10.415851
-0.154659	-1.410000	0.731257	-8.322021	12.351681
-0.160033	-1.470000	0.726073	-8.816704	14.602159
-0.164703	-1.530000	0.721615	-8.998941	17.202920
-0.168757	-1.590000	0.717779	-9.028799	20.190000
-0.172278	-1.650000	0.714475	-8.858096	23.599102
-0.175337	-1.710000	0.711623	-8.431178	27.464767
-0.177999	-1.770000	0.709156	-7.983782	31.810447
-0.180319	-1.830000	0.707016	-6.343014	36.692489
-0.182345	-1.890000	0.705156	-4.927013	42.109025
-0.184118	-1.950000	0.703534	-2.744769	48.068775
-0.185674	-2.010000	0.702116	0.103926	54.684777

TABLE 4. - VALUES OF COORDINATES ALONG CERTAIN STREAMLINES.

$$\tilde{\psi}^{(2)} = 5\tilde{\psi}_{1,1} + \tilde{\psi}_{4,1} + \tilde{\psi}_{3,2}$$

$\tilde{\psi}^{(2)} = 3.37500$

M	θ	V	X	Y
-0.124051	2.010000	0.759889	-1.468293	29.424884
-0.105100	1.890000	0.782051	0.999104	24.269749
-0.079633	1.770000	0.810510	1.787024	19.824884
-0.048576	1.650000	0.847771	2.410300	16.117990
-0.018606	1.530000	0.897734	2.381348	13.134451
0.000000	1.500000	0.912871	2.340366	12.564737
0.010975	1.450000	0.961090	2.296191	11.599273
0.043704	1.390000	0.986618	2.111957	10.602341
0.070496	1.330000	1.027856	1.953007	9.053857
0.099007	1.270000	1.084419	1.812403	9.341831
0.128379	1.210000	1.151768	1.735444	9.117927
0.157063	1.150000	1.230382	1.780358	9.234551
0.182971	1.090000	1.310386	2.059041	9.786227
0.204654	1.030000	1.412364	2.966875	10.860479

$\tilde{\psi}^{(2)} = 1.00000$

M	θ	V	X	Y
-0.188564	2.010000	0.699492	2.324731	31.508271
-0.178327	1.890000	0.708853	4.570726	25.901013
-0.165932	1.770000	0.720448	5.883196	20.980359
-0.150895	1.650000	0.734924	6.480687	16.783773
-0.132648	1.530000	0.753127	6.555356	13.267020
-0.110567	1.410000	0.776165	6.270874	10.309995
-0.084043	1.290000	0.805454	5.761680	8.100360
-0.052634	1.170000	0.842727	5.134936	6.310600
-0.016363	1.050000	0.889895	4.473814	4.979240
0.000000	1.000000	0.912871	4.203723	4.533949
0.016914	0.950000	0.937903	3.942229	4.147762
0.037659	0.890000	0.970619	3.643656	3.754963
0.058469	0.830000	1.006025	3.364623	3.430284
0.078812	0.770000	1.043604	3.106054	3.163602
0.098120	0.710000	1.082542	2.866216	2.944274
0.115875	0.650000	1.121773	2.640637	2.761631
0.131696	0.590000	1.160128	2.422763	2.605958
0.145392	0.530000	1.196515	2.205379	2.469614
0.156962	0.470000	1.230075	1.982281	2.347728
0.166547	0.410000	1.260261	1.749643	2.238222
0.174376	0.350000	1.286831	1.506856	2.141184
0.180716	0.290000	1.309836	1.255469	2.057942
0.185847	0.230000	1.329589	1.000516	1.990091
0.190061	0.170000	1.346670	0.747637	1.938772
0.193669	0.110000	1.361984	0.503340	1.904255
0.197035	0.050000	1.376915	0.274644	1.885795
0.200640	-0.010000	1.393670	0.070254	1.881530

$\tilde{\psi}^{(2)} = 0.12500$

M	θ	V	X	Y
-0.211529	2.010000	0.679175	3.773443	32.257638
-0.205169	1.890000	0.684694	6.087582	26.470435
-0.197529	1.770000	0.691453	7.446596	21.391305
-0.188258	1.650000	0.699770	8.068900	17.000470
-0.177043	1.530000	0.710040	8.147020	13.319372
-0.163525	1.410000	0.722735	7.844661	10.264799
-0.147367	1.290000	0.738388	7.295778	7.795652
-0.128340	1.170000	0.757531	6.405297	5.835603
-0.106478	1.050000	0.780561	5.451252	4.306625
-0.082273	0.930000	0.807476	5.087765	3.160011
-0.056889	0.810000	0.837499	4.348050	2.259640
-0.032202	0.690000	0.868701	3.846916	1.604110
-0.010484	0.570000	0.898018	2.983592	1.118996
0.000000	0.500000	0.912871	2.609678	0.997304
0.006314	0.450000	0.922056	2.346897	0.761997
0.012540	0.390000	0.931295	2.034037	0.622204
0.017311	0.330000	0.938507	1.722694	0.501955
0.020732	0.270000	0.943749	1.411598	0.406608
0.022972	0.210000	0.947214	1.099892	0.332384
0.024255	0.150000	0.949211	0.787037	0.275441
0.024837	0.090000	0.950119	0.472877	0.237553
0.024993	0.030000	0.950364	0.157656	0.218615
0.025007	-0.030000	0.950386	-0.157951	0.218615
0.025164	-0.090000	0.950631	-0.472805	0.237527
0.025750	-0.150000	0.951549	-0.785290	0.275199
0.027062	-0.210000	0.953611	-1.093312	0.331235
0.029415	-0.270000	0.957332	-1.394300	0.408866
0.033168	-0.330000	0.963333	-1.685197	0.494809
0.038755	-0.390000	0.972417	-1.962382	0.590076
0.046744	-0.450000	0.985732	-2.221405	0.714649
0.057933	-0.510000	1.005076	-2.456220	0.834740
0.073579	-0.570000	1.033627	-2.656870	0.954712
0.096024	-0.630000	1.078141	-2.800474	1.054392
0.131984	-0.690000	1.160861	-2.796089	1.048646
0.192680	-0.716946	1.357717	-1.926936	0.294031

TABLE 4. - VALUES OF COORDINATES ALONG CERTAIN STREAMLINES.

$\tilde{\psi}^{(3)} = 5\tilde{\psi}_{1,1} + \tilde{\psi}_{4,1} + \tilde{\psi}_{4,2}$

M	θ	V	X	Y
-0.279488	2.010000	0.623942	-6.319456	37.304800
-0.227505	1.890000	0.665563	-3.078461	29.231646
-0.168590	1.770000	0.717937	-1.310090	22.635248
-0.100653	1.650000	0.786896	-0.572946	17.493515
-0.021572	1.530000	0.882817	-0.486053	13.839625
0.000000	1.500000	0.912871	-0.522606	13.167661
0.037200	1.450000	0.969869	-0.601692	12.325764
0.082540	1.390000	1.050952	-0.673592	11.829271
0.125782	1.330000	1.145372	-0.620835	12.028891
0.162883	1.270000	1.248440	-0.337757	13.049726
0.191581	1.210000	1.353041	0.327928	14.964675

$\tilde{\psi}^{(3)} = 1.00000$

M	θ	V	X	Y
-0.348709	2.010000	0.560033	0.252313	42.709037
-0.334635	1.890000	0.583722	3.853989	31.728749
-0.298655	1.770000	0.609536	5.871999	28.109442
-0.260359	1.650000	0.638808	6.751248	20.024064
-0.219285	1.530000	0.672515	6.858241	15.114722
-0.174977	1.410000	0.711958	6.484232	11.312738
-0.127121	1.290000	0.758786	5.851750	8.495843
-0.075863	1.170000	0.814876	5.123468	6.381508
-0.022355	1.050000	0.881761	4.411277	4.934518
0.000000	1.000000	0.912871	4.138594	4.481473
0.021982	0.950000	0.945679	3.879132	4.101107
0.047304	0.890000	0.986834	3.593952	3.725792
0.071203	0.830000	1.029169	3.333072	3.422182
0.092912	0.770000	1.071688	3.091888	3.173460
0.112200	0.710000	1.113349	2.863855	2.964983
0.128950	0.650000	1.153211	2.641703	2.785192
0.143228	0.590000	1.190543	2.419032	2.626153
0.155226	0.530000	1.224855	2.191260	2.483320
0.165197	0.470000	1.255863	1.956019	2.354794
0.173406	0.410000	1.283434	1.713058	2.240402
0.180101	0.350000	1.307541	1.463853	2.140848
0.185502	0.290000	1.328224	1.211012	2.057029
0.189796	0.230000	1.345572	0.957646	1.980588
0.193145	0.170000	1.359720	0.706792	1.938687
0.195688	0.110000	1.370862	0.460925	1.903985
0.197551	0.050000	1.379266	0.221622	1.884746
0.198851	-0.010000	1.385251	-0.010001	1.880055
0.199694	-0.070000	1.389193	-0.236217	1.889050
0.200073	-0.296981	1.390979	-1.052122	2.039078
0.198364	-0.416981	1.383005	-1.481279	2.199402
0.194599	-0.536981	1.366045	-1.908192	2.420250
0.187121	-0.656981	1.334663	-2.307328	2.691303
0.172475	-0.776981	1.280204	-2.622735	2.964641
0.138739	-0.896981	1.170420	-2.798032	3.110153
0.034489	-0.990000	0.965464	-2.838119	2.938809
0.000000	-1.000000	0.912871	-2.631329	2.928563
-0.084443	-1.018378	0.804779	-2.719093	3.070183
-0.144443	-1.032044	0.741278	-2.872306	3.321934
-0.204443	-1.048360	0.685347	-3.103185	3.715844
-0.264443	-1.067999	0.635591	-3.417928	4.275613
-0.324443	-1.091251	0.590878	-3.823809	5.035206
-0.384443	-1.118238	0.550383	-4.320905	6.040533
-0.444443	-1.149009	0.513481	-4.900935	7.351966
-0.504443	-1.183571	0.479685	-5.605681	9.047627
-0.564443	-1.221904	0.448600	-6.404808	11.227412
-0.624443	-1.263961	0.419934	-7.451327	14.017522
-0.684443	-1.309665	0.393404	-8.486591	17.575316
-0.744443	-1.358912	0.368799	-9.571834	22.094943
-0.804443	-1.411574	0.345932	-10.639123	27.012251
-0.864443	-1.467496	0.324640	-11.781706	35.009367
-0.924443	-1.526497	0.304801	-12.237764	44.017601
-0.982958	-1.586815	0.286720	-12.379838	54.911380

$\tilde{\psi}^{(3)} = 0.00250$

M	θ	V	X	Y
-0.388626	2.010000	0.547701	1.872772	43.923545
-0.358333	1.890000	0.567531	5.363452	34.720008
-0.327082	1.770000	0.589014	7.641734	26.959503
-0.294741	1.650000	0.612440	8.553051	20.559295
-0.261200	1.530000	0.638143	8.604421	15.420648
-0.226419	1.410000	0.666475	8.267035	11.368317
-0.190513	1.290000	0.697733	7.581811	8.297100
-0.153890	1.170000	0.732004	6.767002	5.979659
-0.117428	1.050000	0.768883	5.926074	4.275776
-0.082611	0.930000	0.807090	5.120572	3.040228
-0.051448	0.810000	0.844196	4.371150	2.147893
-0.025979	0.690000	0.876912	3.675998	1.497614
-0.007500	0.570000	0.902196	3.019285	1.017392
0.000000	0.500000	0.912871	2.646735	0.794411
0.004007	0.450000	0.918677	2.383186	0.660819
0.007516	0.390000	0.923825	2.068275	0.520121
0.009845	0.330000	0.927272	1.753017	0.401635
0.011259	0.270000	0.929380	1.438215	0.304038
0.012022	0.210000	0.930519	1.121380	0.226486
0.012365	0.150000	0.931034	0.802809	0.168503
0.012480	0.090000	0.931205	0.482513	0.129878
0.012500	0.030000	0.931235	0.160920	0.110558
0.012498	-0.030000	0.931234	-0.161176	0.110559
0.012494	-0.090000	0.931227	-0.482618	0.129866
0.012432	-0.150000	0.931134	-0.801942	0.168366
0.012201	-0.210000	0.930788	-1.117408	0.229762
0.011620	-0.270000	0.929918	-1.427044	0.301521
0.010427	-0.330000	0.928139	-1.728741	0.394828
0.008265	-0.390000	0.924931	-2.020283	0.504529
0.004637	-0.450000	0.919597	-2.299450	0.629153
-0.001153	-0.510000	0.911214	-2.564158	0.764907
-0.010108	-0.570000	0.898542	-2.812770	0.915869
-0.023734	-0.630000	0.879910	-3.044886	1.074609
-0.044342	-0.690000	0.853004	-3.263341	1.244145
-0.075430	-0.750000	0.815380	-3.479192	1.433647
-0.121598	-0.810000	0.764512	-3.722390	1.674983
-0.182606	-0.867004	0.704917	-4.038909	2.024304
-0.242606	-0.913198	0.653065	-4.383331	2.460528
-0.302606	-0.955648	0.606625	-4.831906	3.063879
-0.362606	-0.997329	0.564677	-5.369401	3.861209
-0.422606	-1.039788	0.526529	-6.020541	4.920303
-0.482606	-1.083913	0.491652	-6.797200	6.315539
-0.542606	-1.130250	0.459626	-7.707754	8.140979
-0.602606	-1.179121	0.430110	-8.754030	10.515102
-0.662606	-1.230703	0.402827	-9.927090	13.586581
-0.722606	-1.285073	0.377544	-11.201112	17.530260
-0.782606	-1.342229	0.354064	-12.525146	22.597894
-0.842606	-1.402114	0.332219	-13.812369	29.037962
-0.900245	-1.462114	0.312639	-14.888438	36.807441
-0.955793	-1.522114	0.294964	-15.609801	46.088545
-1.009567	-1.582114	0.278885	-15.803181	57.056381

TABLE 4. - VALUES OF COORDINATES ALONG CERTAIN STREAMLINES.

$$\tilde{\psi}^{(4)} = 5\tilde{\psi}_{1,1} + \tilde{\psi}_{2,1} + \tilde{\psi}_{3,1} + \tilde{\psi}_{4,1} + \tilde{\psi}_{4,2}$$

H	θ	V	X	Y
-0.203388	2.010000	0.686274	-4.471703	47.270346
-0.161799	1.890000	0.724382	-0.760406	37.990946
-0.118854	1.770000	0.760488	1.495280	30.279158
-0.067061	1.650000	0.826262	2.183143	24.006148
-0.014186	1.530000	0.862067	2.409951	19.295070
0.000000	1.500000	0.912871	2.436109	18.324944
0.024272	1.450000	0.949270	2.180963	16.906031
0.054367	1.390000	0.990819	1.995186	15.536006
0.085058	1.330000	1.059021	1.807440	14.553944
0.115620	1.270000	1.121201	1.530470	14.000093
0.144033	1.210000	1.192841	1.318773	13.938661
0.171564	1.150000	1.277870	1.737232	14.458352
0.194252	1.090000	1.304523	2.331941	15.076969

$\tilde{\psi}^{(4)} = 1.00000$

H	θ	V	X	Y
-0.272254	2.010000	0.629505	1.919966	51.851205
-0.242594	1.890000	0.653074	5.989671	41.898525
-0.212256	1.770000	0.678547	8.411574	33.016496
-0.181167	1.650000	0.706236	9.343952	25.764604
-0.149267	1.530000	0.736519	9.471227	19.845782
-0.116528	1.410000	0.769831	9.905190	15.125289
-0.082990	1.290000	0.806656	8.189183	11.447139
-0.048810	1.170000	0.847479	7.404490	8.647730
-0.014327	1.050000	0.892692	6.178112	6.566543
0.000000	1.000000	0.912871	5.750822	5.875300
0.014224	0.950000	0.933829	5.450055	5.279550
0.031067	0.890000	0.959963	4.879417	4.652708
0.047545	0.830000	0.987090	4.431312	4.131415
0.063530	0.770000	1.015078	4.907836	3.694746
0.078890	0.710000	1.043754	3.909125	3.330170
0.093508	0.650000	1.072917	3.434320	3.024062
0.107285	0.590000	1.102339	2.881027	2.774678
0.120149	0.530000	1.131791	2.949638	2.568142
0.132061	0.470000	1.161058	2.435631	2.394380
0.143011	0.410000	1.189949	1.937708	2.253987
0.153018	0.350000	1.218310	1.854363	2.140633
0.162117	0.290000	1.246013	1.383796	2.050848
0.170353	0.230000	1.272942	1.124740	1.981833
0.177770	0.170000	1.298966	0.875845	1.931297
0.184398	0.110000	1.323899	0.635544	1.897368
0.190248	0.050000	1.347444	0.401806	1.879585
0.195301	-0.010000	1.369144	0.171992	1.873965
0.199515	-0.070000	1.388353	-0.057041	1.883133

$\tilde{\psi}^{(4)} = 0.00250$

H	θ	V	X	Y
-0.287824	2.010000	0.617619	3.456795	52.910710
-0.260788	1.890000	0.638669	7.812970	42.541058
-0.233642	1.770000	0.660441	9.998196	33.635636
-0.206452	1.650000	0.683589	11.058365	26.156288
-0.179325	1.530000	0.707931	11.190028	20.009160
-0.152430	1.410000	0.733425	10.700759	15.059304
-0.126018	1.290000	0.759924	9.833554	11.159294
-0.100452	1.170000	0.787117	8.769004	8.129169
-0.076220	1.050000	0.814460	7.633730	5.827637
-0.053939	0.930000	0.841118	6.508587	4.104656
-0.034305	0.810000	0.865960	5.439194	2.831430
-0.017091	0.690000	0.887671	4.445362	1.901812
-0.005481	0.570000	0.905045	3.730247	1.237045
0.000000	0.500000	0.912871	3.030586	0.935515
0.003128	0.450000	0.917397	2.887797	0.750089
0.006091	0.390000	0.921712	2.490608	0.581568
0.008260	0.330000	0.924924	1.907810	0.437297
0.009796	0.270000	0.927200	1.337760	0.327700
0.010834	0.210000	0.928745	1.179226	0.234856
0.011521	0.150000	0.929770	0.831034	0.171409
0.011990	0.090000	0.930472	0.497198	0.130492
0.012345	0.030000	0.931003	0.161956	0.110586
0.012645	-0.030000	0.931453	-0.160211	0.110539
0.012896	-0.090000	0.931830	-0.474549	0.129391
0.013040	-0.150000	0.932046	-0.781132	0.166311
0.012952	-0.210000	0.931914	-1.079684	0.229607
0.012431	-0.270000	0.931133	-1.369795	0.291574
0.011196	-0.330000	0.929285	-1.650052	0.378687
0.008868	-0.390000	0.925822	-1.927092	0.489554
0.004949	-0.450000	0.920054	-2.182664	0.586888
-0.001218	-0.510000	0.911120	-2.431753	0.724535
-0.010516	-0.570000	0.897073	-2.668049	0.868640
-0.024152	-0.630000	0.879351	-2.894512	1.022997
-0.043779	-0.690000	0.853806	-3.111600	1.191518
-0.071579	-0.750000	0.819887	-3.328295	1.381753
-0.110029	-0.810000	0.776741	-3.562584	1.614040
-0.160829	-0.870000	0.725309	-3.846368	1.931814
-0.221239	-0.928365	0.670853	-4.209398	2.390525
-0.281239	-0.980506	0.622607	-4.631948	2.989140
-0.341239	-1.030780	0.579148	-5.136570	3.787427
-0.401239	-1.081293	0.539716	-5.736652	4.851608
-0.461239	-1.133245	0.503726	-6.440548	6.264262
-0.521239	-1.187345	0.470727	-7.249988	8.130452
-0.581239	-1.244004	0.440352	-8.156092	10.589375
-0.641239	-1.303428	0.412302	-9.133158	13.789935
-0.699115	-1.363428	0.387217	-10.095364	17.789192
-0.754637	-1.423428	0.364796	-10.964291	22.676375
-0.808142	-1.483428	0.344576	-11.651981	28.560715
-0.859927	-1.543428	0.326196	-12.048281	35.564312
-0.910246	-1.603428	0.309373	-12.017554	43.929630
-0.959313	-1.663428	0.293881	-11.395326	53.692414
-1.007306	-1.723428	0.279542	-9.984841	65.033651

TABLE 4. - VALUES OF COORDINATES ALONG CERTAIN STREAMLINES.

$\tilde{\psi}^{(5)} = 5\tilde{\psi}_{1,1} + \tilde{\psi}_{5,1} + \tilde{\psi}_{5,2}$

$\tilde{\psi}^{(5)} = 3.37500$

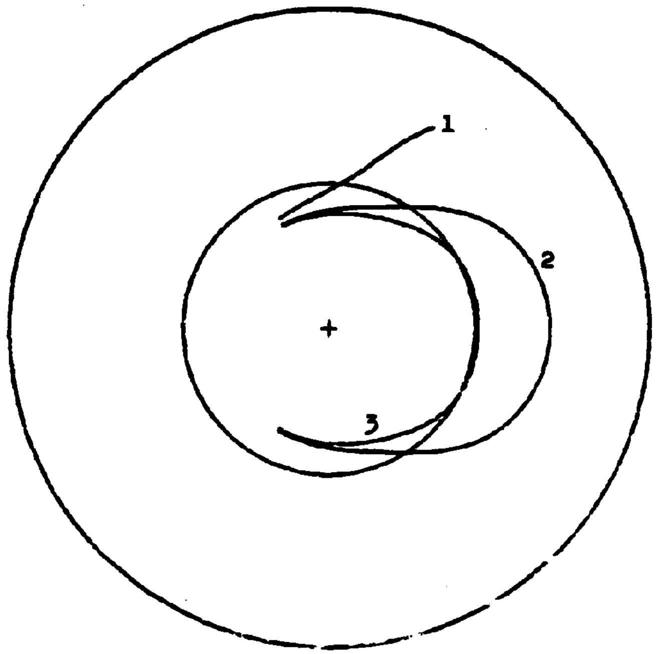
H	θ	V	X	Y
-0.246358	2.010000	0.634097	-0.737851	22.378451
-0.208843	1.890000	0.681505	0.785607	18.774131
-0.148230	1.770000	0.737538	1.538641	15.608678
-0.084291	1.650000	0.805171	1.803748	13.094650
-0.017177	1.530000	0.858783	1.948568	11.091344
0.000000	1.500000	0.912871	1.928113	10.685987
0.020831	1.450000	0.956407	1.870603	10.161971
0.063459	1.390000	1.014950	1.790061	9.584578
0.097513	1.330000	1.081282	1.719556	9.229981
0.130067	1.270000	1.156011	1.804280	9.139107
0.159026	1.210000	1.230160	1.763475	9.327432
0.185841	1.150000	1.329505	1.998218	9.690156

$\tilde{\psi}^{(5)} = 1.00000$

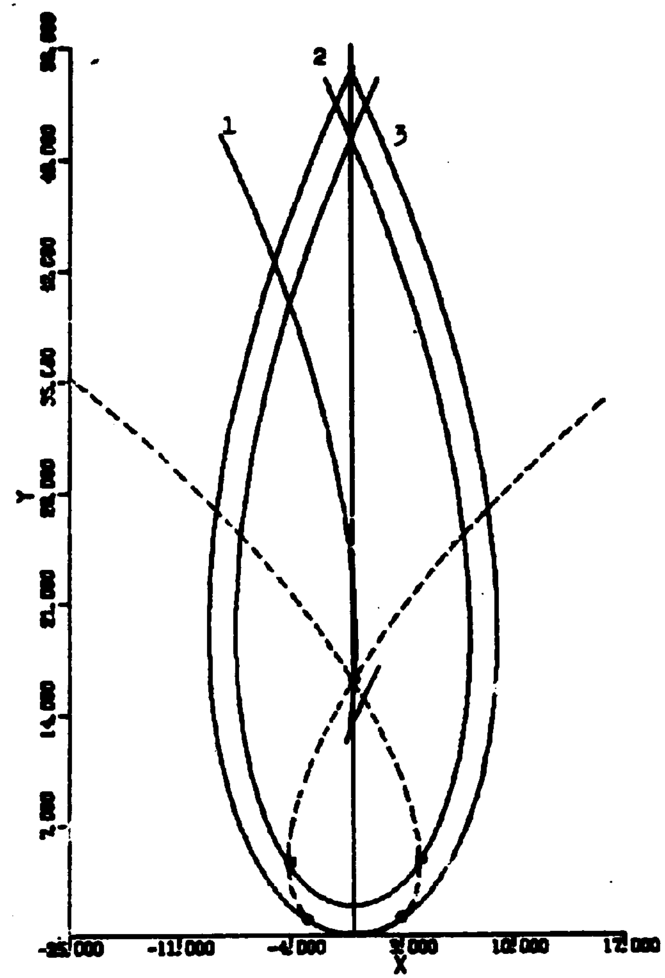
H	θ	V	X	Y
-0.388784	2.010000	0.547651	3.905142	25.780934
-0.348084	1.890000	0.576505	4.694096	21.542383
-0.308814	1.770000	0.608165	5.497363	17.804487
-0.254970	1.650000	0.643067	6.161936	14.530685
-0.206490	1.530000	0.681804	6.220057	11.741612
-0.161281	1.410000	0.724915	5.990345	9.419380
-0.113492	1.290000	0.773046	5.370752	7.531917
-0.065784	1.170000	0.826755	5.044130	6.036689
-0.018466	1.050000	0.886384	4.475009	4.883730
0.000000	1.000000	0.912871	4.436900	4.491393
0.018497	0.950000	0.940317	4.402240	4.144783
0.039913	0.890000	0.974322	3.727311	3.783229
0.060787	0.830000	1.009256	3.461508	3.474081
0.079423	0.770000	1.044782	3.205414	3.210077
0.097108	0.710000	1.080495	2.958396	2.984282
0.113327	0.650000	1.115915	2.718720	2.790283
0.127874	0.590000	1.150536	2.483818	2.622457
0.140784	0.530000	1.183848	2.250492	2.474219
0.152016	0.470000	1.215370	2.016370	2.348156
0.161782	0.410000	1.244705	1.778356	2.236080
0.169925	0.350000	1.271492	1.535036	2.138885
0.176008	0.290000	1.295484	1.285949	2.054335
0.182484	0.230000	1.316521	1.031897	1.980742
0.187087	0.170000	1.334520	0.774704	1.936609
0.190748	0.110000	1.349535	0.517494	1.900317
0.193600	0.050000	1.361605	0.261153	1.870889
0.195775	-0.010000	1.371247	0.015140	1.874875
0.197418	-0.070000	1.378459	-0.223754	1.884375
0.198696	-0.130000	1.384534	-0.451384	1.907147
0.199006	-0.190000	1.389732	-0.664257	1.941752

$\tilde{\psi}^{(5)} = 0.12500$

H	θ	V	X	Y
-0.432104	2.010000	0.520801	4.300914	27.081471
-0.393150	1.890000	0.544818	6.282361	22.010927
-0.353921	1.770000	0.570499	7.348487	18.016061
-0.314503	1.650000	0.597975	7.846860	15.109535
-0.275035	1.530000	0.627358	7.909986	12.072522
-0.235737	1.410000	0.658708	7.655292	9.509212
-0.196931	1.290000	0.691985	7.161505	7.381102
-0.159074	1.170000	0.726903	6.309988	5.646479
-0.122786	1.050000	0.763274	5.883136	4.256731
-0.088845	0.930000	0.800012	5.168506	3.166837
-0.058151	0.810000	0.835959	4.449285	2.307616
-0.031630	0.690000	0.869450	3.740671	1.651178
-0.010074	0.570000	0.898590	3.063653	1.151893
0.000000	0.500000	0.912871	2.873838	0.920669
0.000041	0.450000	0.921654	2.398795	0.770157
0.012036	0.390000	0.930542	2.071996	0.633129
0.016717	0.330000	0.937603	1.748245	0.511197
0.020176	0.270000	0.942893	1.427027	0.411777
0.022546	0.210000	0.946553	1.107844	0.339626
0.023997	0.150000	0.948808	0.790208	0.278798
0.024731	0.090000	0.949954	0.473042	0.237605
0.024980	0.030000	0.950343	0.157609	0.218616
0.024992	-0.030000	0.950363	-0.155001	0.218815
0.025027	-0.090000	0.950416	-0.473737	0.237590
0.025337	-0.150000	0.950903	-0.789637	0.275696
0.026167	-0.210000	0.952203	-1.105648	0.333223
0.027735	-0.270000	0.954672	-1.421487	0.410544
0.030230	-0.330000	0.958629	-1.736863	0.506058
0.033809	-0.390000	0.964365	-2.050338	0.624172
0.038589	-0.450000	0.972144	-2.361304	0.765114
0.044656	-0.510000	0.982214	-2.668274	0.928921
0.052062	-0.570000	0.994821	-2.969019	1.109227
0.060835	-0.630000	1.010234	-3.261080	1.305056
0.070987	-0.690000	1.028767	-3.541246	1.522490
0.082527	-0.750000	1.050828	-3.805461	1.754184
0.095473	-0.810000	1.076991	-4.048529	1.998335
0.109879	-0.870000	1.108113	-4.263538	2.234180
0.125878	-0.930000	1.145607	-4.440995	2.454851
0.143784	-0.990000	1.192089	-4.563502	2.631600
0.164405	-1.050000	1.253306	-4.599181	2.687280
0.190836	-1.110000	1.350027	-4.428566	2.358972



HODOGRAPH PLANE



PHYSICAL PLANE

Fig. 1 - $\tilde{\psi}^{(1)} = 5\tilde{\psi}_{1,1} + \tilde{\psi}_{5,1} + \tilde{\psi}_{4,2}$

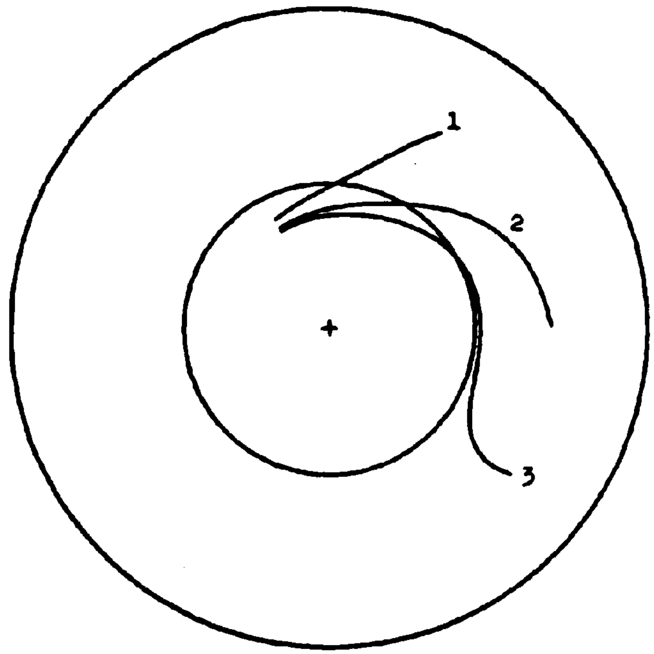
1 $\tilde{\psi}^{(1)} = 5.0625$

2 $\tilde{\psi}^{(1)} = 1.0000$

3 $\tilde{\psi}^{(1)} = 0.0625$

----- Sonic Line

● Transition Points



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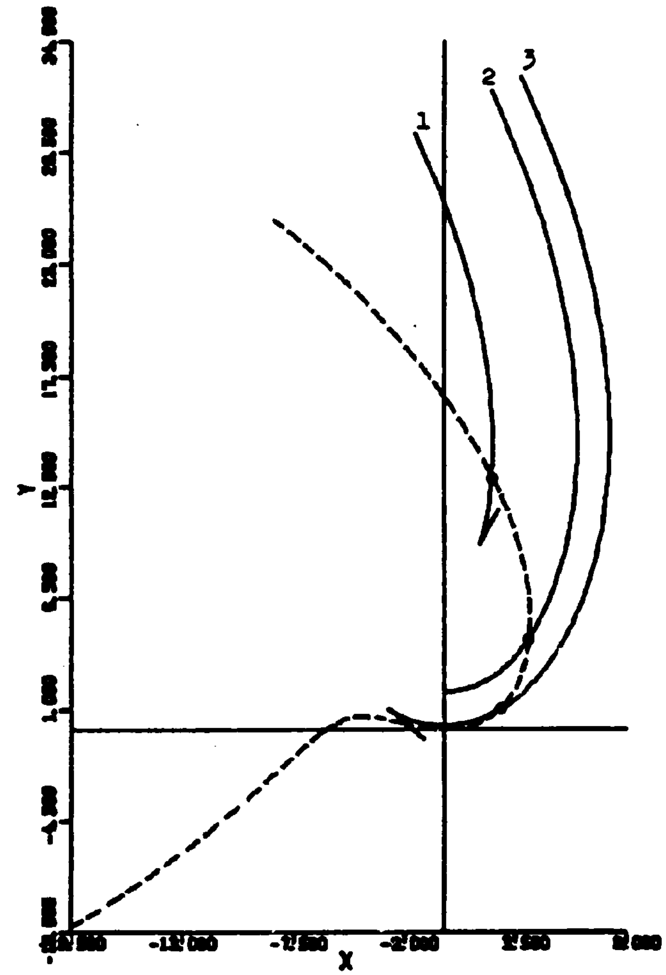


PHYSICAL PLANE

Fig. 2 - $\tilde{\psi}^{(2)} = 5\tilde{\psi}_{1,1} + \tilde{\psi}_{4,1} + \tilde{\psi}_{3,2}$

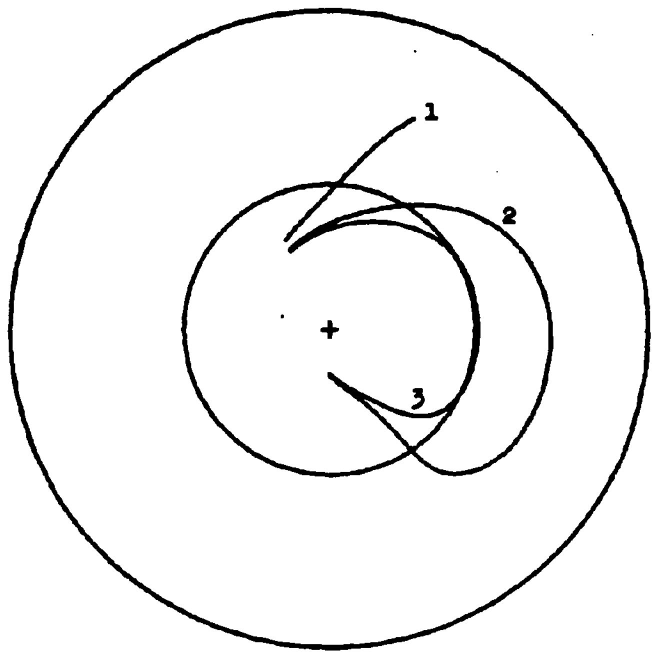
1 $\tilde{\psi}^{(2)} = 3.375$

2 $\tilde{\psi}^{(2)} = 1.000$

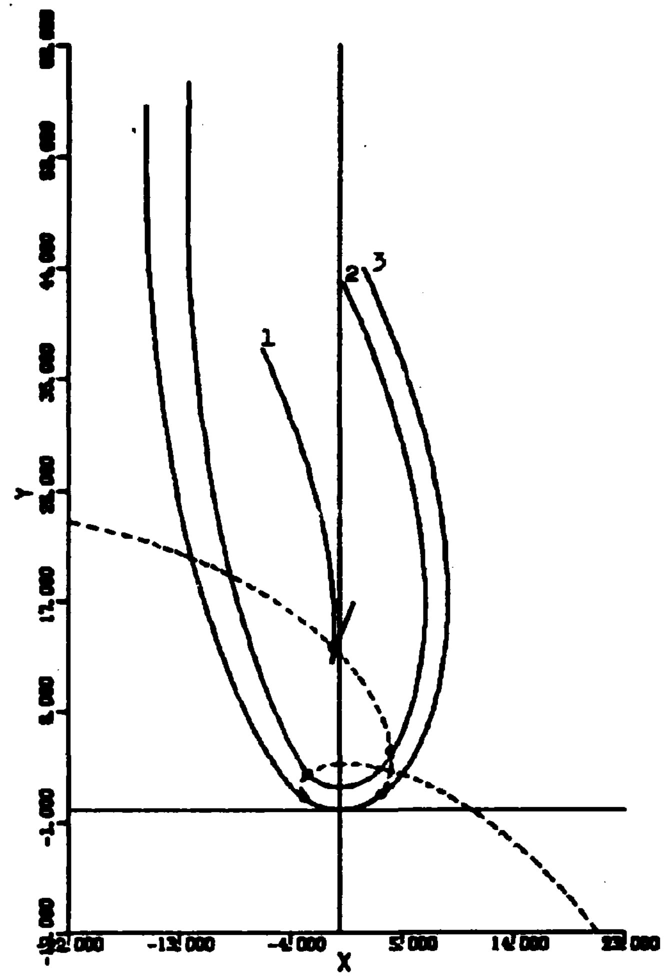
3 $\tilde{\psi}^{(2)} = 0.125$

----- Sonic Line

● Transition Points



HODOGRAPH PLANE



PHYSICAL PLANE

Fig. 3 - $\tilde{\psi}^{(3)} = 5\tilde{\psi}_{1,1} + \tilde{\psi}_{4,1} + \tilde{\psi}_{4,2}$

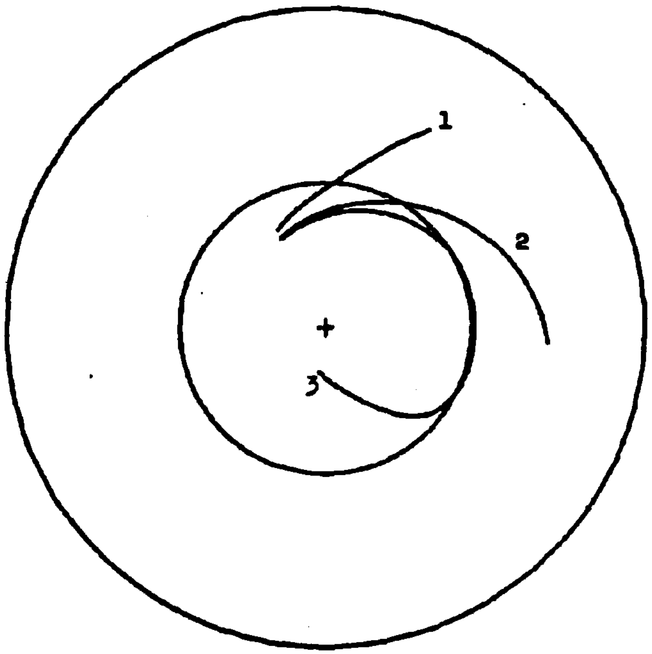
1 $\tilde{\psi}^{(3)} = 5.0625$

2 $\tilde{\psi}^{(3)} = 1.0000$

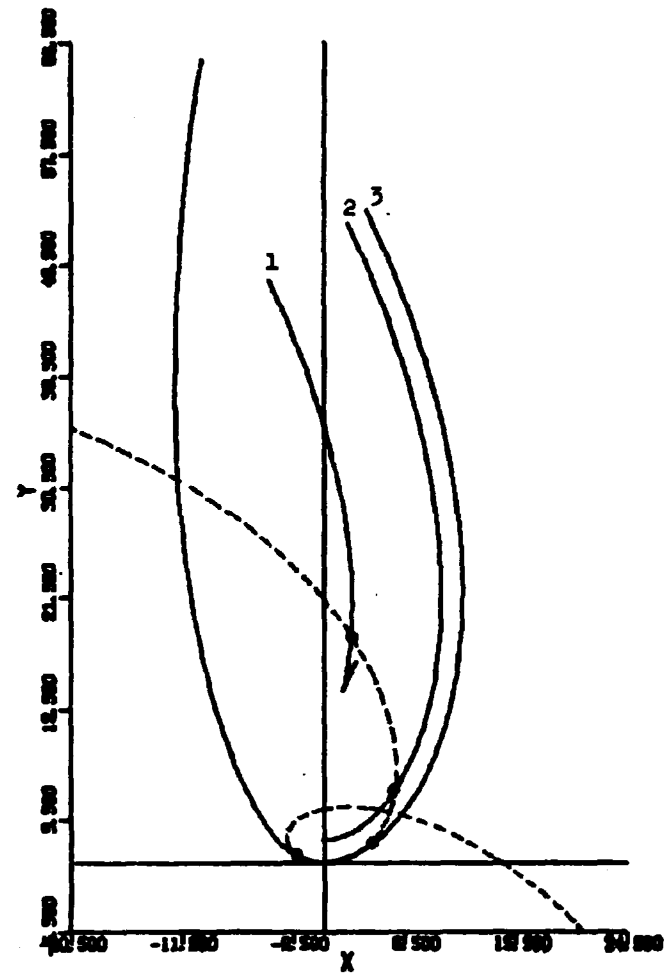
3 $\tilde{\psi}^{(3)} = 0.0625$

----- Sonic Line

• Transition Points



HODOGRAPH PLANE



PHYSICAL PLANE

Fig. 4 - $\tilde{\psi}^{(4)} = 5\tilde{\psi}_{1,1} + \tilde{\psi}_{2,1} + \tilde{\psi}_{3,1} + \tilde{\psi}_{4,1} + \tilde{\psi}_{4,2}$

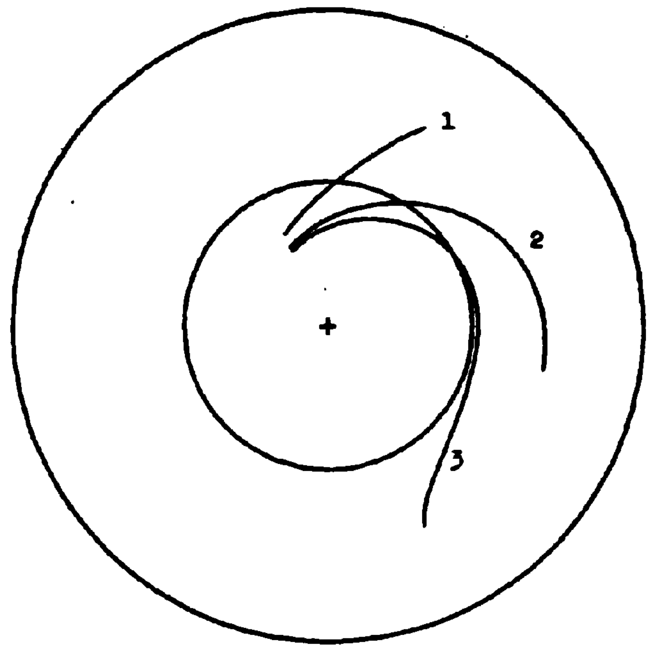
1. $\tilde{\psi}^{(4)} = 5.0625$

2 $\tilde{\psi}^{(4)} = 1.0000$

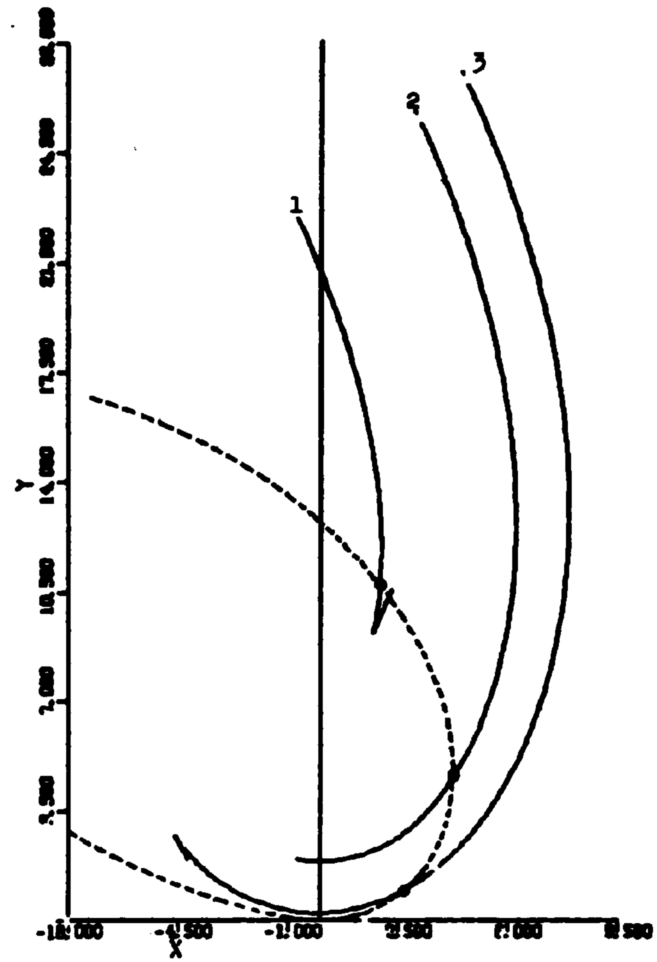
3 $\tilde{\psi}^{(4)} = 0.0625$

----- Sonic Line

● Transition Points



HODOGRAPH PLANE



PHYSICAL PLANE

Fig. 5 - $\tilde{\psi}^{(5)} = 5\tilde{\psi}_{1,1} + \tilde{\psi}_{3,1} + \tilde{\psi}_{3,2}$

1 $\tilde{\psi}^{(5)} = 3.375$

2 $\tilde{\psi}^{(5)} = 1.000$

3 $\tilde{\psi}^{(5)} = 0.125$

----- Sonic Line

● Transition Points

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