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THE CALCULATION OF ZEROS OF POLYNOMIALS AND ANALYTIC FUNCTIONS

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TECHNICAL REPORT 36
APRIL 12, 1966

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Presented at a Symposium on Mathematical Aspects of Computer Science sponsored by the American Mathematical Society, April 1966, New York City.

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1. INTRODUCTION

We study a class of new methods for the calculation of zeros. In Sections 2 to 8 we treat the case of a polynomial with all distinct zeros and one zero of largest modulus. We studied this case in detail in [16]. Here we give a simplified treatment and also obtain some new results. In Sections 9 and 10 we treat the case of a zero of smallest modulus.

In the remaining sections we discuss the calculation of multiple zeros and equimodular dominant zeros of polynomials and zeros of analytic functions. Detailed analysis of these matters as well as material concerning the calculation of subdominant zeros will appear elsewhere.

2. DESCRIPTION OF THE BASIC ALGORITHM FOR THE DOMINANT ZERO OF A POLYNOMIAL

Let

$$P(t) = \sum_{j=0}^{n} a_j t^{n-j} \ , \ a_0 = 1 \tag{2.1}$$

be a polynomial with complex coefficients and with zeros $\rho_1, \rho_2, \ldots, \rho_n$. In Sections 2 to 8 we assume the zeros are distinct and $|\rho_1| > |\rho_i|$, $i > 1$. We generate a sequence of polynomials as follows. Let $B(t)$ be an arbitrary polynomial of degree at most $n-1$ such that $B(\rho_1) \neq 0$. Define

$$\begin{aligned} G(0,t) &= B(t) \ , \\ G(\lambda+1,t) &= tG(\lambda,t) - \alpha_0(\lambda)P(t) \ , \end{aligned} \tag{2.2}$$

where $\alpha_0(\lambda)$ is the leading coefficient of $G(\lambda,t)$. Then all the $G(\lambda,t)$ are polynomials of degree at most $n-1$.

We generate the $G(\lambda,t)$ until we have calculated, say, $G(\Lambda,t)$. We use $G(\Lambda,t)$ to construct an iteration function. (In the remainder of this paper we do not distinguish between the running index λ and a fixed value of λ equal to Λ.) We choose an initial approximation t_o and generate a sequence $\{t_i\}$ by

$$t_{i+1} = \varphi(\lambda, t_i) \tag{2.3}$$

where

$$\varphi(\lambda,t) = t - \frac{\alpha_o(\lambda)P(t)}{G(\lambda,t)} \quad . \tag{2.4}$$

The t_i form the approximating sequence for ρ_1.

We have described a two-stage algorithm.

a. Preprocessing stage: This is specified by the recursion for the G polynomials given by (2.2).

b. Iteration stage: This is specified by (2.3) and (2.4).

3. A NUMERICAL EXAMPLE

For illustration we calculate the dominant zero of

$$P(t) = (t+1)(t-2)(t+3) = t^3 + 2t^2 - 5t - 6 \quad .$$

We choose

$$G(0,t) = t^3 - P(t) = -2t^2 + 5t + 6 \quad .$$

(The reason for this choice of $G(0,t)$ is explained in Section 4.) Then

$$G(1,t) = 9t^2 - 4t - 12$$

$$\vdots$$

$$G(9,t) = 53417t^2 - 52052t - 105468 \quad .$$

We now iterate using

$$\varphi(9,t) = t - P(t)\frac{\alpha_o(9)}{G(9,t)} \quad ,$$

and choosing $t_o = 100000$ as our initial approximation. We calculate the sequence of approximations exhibited in Table 1. The sequence is converging alternatingly towards the zero at -3 which is the largest zero in modulus. In the righthand column we exhibit the ratios of successive errors. After the first iteration these ratios are constant. This is as expected because the method used here is first order. (The extension to higher order is described in Section 4.) Observe that all the ratios are small and that the initial ratio is particularly small. These facts are characteristic of the method and are quantitatively explained in Section 7.

TABLE 1. SEQUENCE OF APPROXIMANTS

i	t_i	$(t_{i+1}-\rho_1)/(t_i-\rho_1)$
0	100000.	
1	-2.97	2.6×10^{-7}
2	-3.0001	-5.2×10^{-3}
3	-2.9999993	-5.1×10^{-3}
4	-3.000000003	-5.1×10^{-3}
5	-2.99999999998	-5.1×10^{-3}
6	-3.00000000000009	-5.1×10^{-3}

$\rho_1 = -3$

Note that the rate of convergence of the iteration "looks" numerically quadratic over the entire range of the iteration even though it is asymptotically a first order process. The explanation for this lies in that the error at each step is the product of two small errors, one of which is the error at the previous step. See Section 7. This should be contrasted with the behavior of, say, the Newton-Raphson iteration which is asymptotically quadratic but which behaves linearly when the approximations are far from the zeros. (The reader is referred to Forsythe [6] for an example of this.)

4. COMMENTS ON AND EXTENSIONS OF THE BASIC ALGORITHM

Note that the recursion for the G polynomials defined by (2.2) is easily performed by hand or machine. The multiplication by t is only a shift. All that is then required is a scalar-vector multiplication at each step. Another method for generating the $G(\lambda,t)$ which calculates $G(2\lambda,t)$ directly from $G(\lambda,t)$, $G(\lambda+1,t),\ldots, G(\lambda+n-1,t)$ is described in Traub [16, pp. 126-129].

From (2.2) it follows that $\varphi(\lambda,t)$, which is defined by

$$\varphi(\lambda,t) = t - P(t)\frac{\alpha_o(\lambda)}{G(\lambda,t)} \quad , \tag{4.1}$$

may also be written as

$$\varphi(\lambda,t) = \frac{G(\lambda+1,t)}{G(\lambda,t)} \quad . \tag{4.2}$$

Since, as we verify in Section 6, $\alpha_o(\lambda)$ does not vanish for λ sufficiently large, (4.2) exhibits the iteration function as the ratio of polynomials of degree exactly n-1 . This form is used when t is large. Equation (4.1) exhibits $\varphi(\lambda,t)$ in incremental form.

It may be shown that if any of the zeros of P have magnitude greater than unity, then the coefficients of $G(\lambda,t)$ increase without limit. On the other hand, if all the zeros lie within the unit circle, $G(\lambda,t)$ converges to the zero polynomial. This difficulty is taken care of as follows: Let $\bar{h}(t)$ denote a polynomial $h(t)$ divided by its leading coefficient. We show in Section 6 that

$$\lim_{\lambda\to\infty} \bar{G}(\lambda,t) = \frac{P(t)}{t-\rho_1} \quad .$$

Hence $\bar{G}(\lambda,t)$ has well-behaved coefficients. The $\bar{G}(\lambda,t)$ satisfy the recursion

$$\begin{aligned} &\bar{G}(\lambda+1,t) = \overline{t\bar{G}(\lambda,t) - P(t)}, \quad \text{if} \quad \alpha_o(\lambda) \neq o \\ &\bar{G}(\lambda+1,t) = t\bar{G}(\lambda,t) \quad \text{if} \quad \alpha_o(\lambda) = o \quad . \end{aligned} \tag{4.3}$$

We can write the iteration function as

$$\varphi(\lambda,t) = t - \frac{P(t)}{\bar{G}(\lambda,t)} \quad . \tag{4.4}$$

We turn to the question of choosing the arbitrary polynomial $B(t)$ that appears in (2.2). Recall that $B(t)$ can be any polynomial of degree at most n-1 such that $B(\rho_1) \neq 0$. Two natural choices

for $B(t)$ are $B(t) = P'(t)$ and $B(t) = 1$. If $B(t) = G(0,t) = 1$, it is easy to show that $G(n,t) = t^n - P(t)$. Hence we might as well take $B(t) = G(0,t) = t^n - P(t)$ and this was done in the numerical example of Section 3. Additional discussion of the choice of $B(t)$ may be found in Section 11.

The iteration function $\varphi(\lambda,t)$ is first order. From $G(\lambda,t)$ and its derivatives and $P(t)$ and its derivatives one may construct iteration functions of arbitrarily high order. A general treatment is presented in Traub [16, pp. 116-119].

Because of the rapidity of convergence of this type of method we would generally not use an iteration function of order greater the two. The second order iteration function is given by

$$\varphi_2(t) = t - \frac{P(t)G(\lambda,t)}{P'(t)G(\lambda,t) - P(t)G'(\lambda,t)} \quad .$$

We give a simple numerical example of a second order iteration. Let $P(t) = t^4 - 46t^3 + 528t^2 - 1090t + 2175$. The zeros are $\rho_1 = 29$, $\rho_2 = 15$, $\rho_{3,4} = 1 \pm 2i$. We take $B(t) = 1$, $\lambda = 16$ and choose our initial approximation as $t_o = 100000$. We calculate

$$t_1 = 28.9996$$

$$t_2 = 28.999999999997 \quad .$$

The other iteration functions discussed in later sections of this paper could also be made of arbitrary order. For the sake of simplicity of exposition we shall confine ourselves to the first order case.

5. GLOBAL CONVERGENCE

We state without proof the theorem of global convergence for the iteration functions $\varphi(\lambda,t)$. A proof of this theorem in a form which covers the extension to iteration functions of arbitrary order may be found in Traub [16, pp. 121-122].

THEOREM. Let the zeros ρ_i of the polynomial P be distinct with $|\rho_1| > |\rho_i|$, $i = 2,3,\ldots, n$. Let t_o be an arbitrary point in the extended complex plane such that $t_o \neq \rho_2, \rho_3, \ldots, \rho_n$ and let $t_{i+1} = \varphi(\lambda,t_i)$. Then for all sufficiently large but fixed λ, the sequence t_i is defined for all i and $t_i \to \rho_1$.

The phrase "global convergence" is used in the following sense. For any polynomial whose zeros are distinct and which possesses a largest zero and for any choice of t_o which does not coincide with a subdominant zero, we can conclude that for all sufficiently large λ the sequence t_i defined by $t_{i+1} = \varphi(\lambda,t_i)$ exists and converges to ρ_1. The size of λ depends on P and t_o. It is determined primarily by the ratio of the magnitude of the largest subdominant zero to the magnitude of the dominant zero.

6. PROPERTIES OF THE G POLYNOMIALS

We obtain the principle properties of the G polynomials from the defining recursion

$$G(0,t) = B(t)$$

$$G(\lambda + 1,t) = G(\lambda,t) - \alpha_0(\lambda)\, P(t), \tag{6.1}$$

where $\alpha_0(\lambda)$ is the leading coefficient of $G(\lambda,t)$.

The G polynomials can be introduced in a number of different ways. In [16, p. 114], we define $G(\lambda,t)$ as the remainder of the division of $B(t)t^{\lambda}$ by $P(t)$. The G polynomials can also be defined as the sequence generated by a Bernoulli recurrence with initial conditions which depend on the choice of $B(t)$.

From (6.1) it follows that $G(\lambda + 1, \rho_i) = \rho_i G(\lambda,\rho_i)$.

Hence

$$G(\lambda,\rho_i) = \rho_i^{\lambda} G(0,\rho_i) = \rho_i^{\lambda} B(\rho_i) \quad . \tag{6.2}$$

Since $G(\lambda,t)$ is a polynomial of degree at most n-1, we conclude from Langrange's interpolation formula that

$$G(\lambda,t) = \sum_{i=1}^{n} c_i \rho_i^{\lambda} \frac{P(t)}{t-\rho_i} , \quad c_i = \frac{B(\rho_i)}{P'(\rho_i)} \quad . \tag{6.3}$$

Since $B(\rho_1) \neq 0$ by hypothesis, $c_1 \neq 0$.

Let $\beta(\lambda)$ be the weighted power sum

$$\beta(\lambda) = \sum_{i=1}^{n} c_i \rho_i^{\lambda} \quad . \tag{6.4}$$

From (6.3)

$$\alpha_o(\lambda) = \beta(\lambda) \quad . \tag{6.5}$$

Hence for λ sufficiently large, $\alpha_o(\lambda) \neq 0$.

From (6.3), (6.4) and (6.5) we obtain immediately the most important property of $G(\lambda,t)$, namely

$$\lim_{\lambda\to\infty} \tilde{G}(\lambda,t) = \lim_{\lambda\to\infty} \frac{G(\lambda,t)}{\alpha_o(\lambda)} = \frac{P(t)}{t - \rho_1} , \tag{6.6}$$

for all finite t.

Furthermore the rate of convergence depends on the ratio of the magnitude of the largest subdominant zero to the magnitude of the dominant zero.

To see the importance of (6.6), consider a general iteration function,

$$\psi(t) = t - \frac{P(t)}{V(t)}$$

where $V(t)$ is some function which is yet to be specified. If

(6.7) $$V(t) = \frac{P(t)}{t-\rho_1}$$

then $\psi(t) = \rho_1$ and we always obtain the answer in one step. In the Newton-Raphson method, $V(t) = P'(t)$ and (6.7) is satisfied only at $t = \rho_1$. Equation (6.6) shows that when $V(t) = \bar{G}(\lambda,t)$, then (6.7) is satisfied for all finite t as λ goes to infinity and is satisfied arbitrarily closely for λ sufficiently large.

We obtain an interesting interpretation of the recursion for the G polynomials by considering the Laurent expansion of $G(\lambda,t)/P(t)$. Let

(6.8) $$\frac{G(\lambda,t)}{P(t)} = \sum_{k=0}^{\infty} \frac{d_k(\lambda)}{t^{k+1}} \quad .$$

Clearly, $d_o(\lambda) = \alpha_o(\lambda) = \beta(\lambda)$. Write the recurrence for $G(\lambda,t)$ as

(6.9) $$\frac{G(\lambda+1,t)}{P(t)} = \frac{tG(\lambda,t)}{P(t)} - \alpha_o(\lambda) \quad .$$

Then we conclude that

(6.10) $$d_{k+1}(\lambda) = d_k(\lambda+1) \quad .$$

Hence the right side of (6.9) may be viewed as the operation of performing a left shift upon the vector of coefficients of the Laurent expansion.

From (6.10),

$$d_k(\lambda) = d_o(\lambda+k) = \beta(\lambda+k) \quad ,$$

a result which could also have been obtained directly from the partial fraction expansion of $G(\lambda,t)/P(t)$.

Hence

$$\frac{G(\lambda,t)}{P(t)} = t^{\lambda}\left[\frac{B(t)}{P(t)} - \sum_{k=0}^{\lambda-1} \frac{\beta(k)}{t^{k+1}}\right] \quad . \tag{6.11}$$

Thus, except for a factor of t^{λ} , $G(\lambda,t)/P(t)$ is just the remainder of the series for $G(0,t)/P(t)$ after λ terms.

Finally we mention that the recursion for the G polynomials may be cast as a matrix-vector multiplication where the matrix is the companion matrix of P . We do not pursue this here. The interested reader is referred to the papers by Bauer in the bibliography.

7. THE BEHAVIOR OF THE ERROR

In the numerical example of Section 3 we noted that the ratios of successive errors were small, and that the initial ratio was particularly small when t_o was large. We now study the behavior of the error quantitatively.

Let

$$E(\lambda,t) = \frac{\varphi(\lambda,t)-\rho_1}{t-\rho_1} \quad .$$

From (4.2) and (6.3),

$$E(\lambda,t) = \frac{\sum_{i=2}^{n} d_i(\rho_i/\rho_1)^{\lambda}\frac{(\rho_i-\rho_1)}{t-\rho_i}}{1 + \sum_{i=2}^{n} d_i(\rho_i/\rho_1)^{\lambda}\frac{(t-\rho_1)}{t-\rho_i}} \quad , \quad d_i = c_i/c_1 \tag{7.1}$$

This result is exact. We draw a number of conclusions.

$E(\lambda,t)$ is of order $(\rho_2/\rho_1)^\lambda$ and can be made arbitrarily small.

For the remainder of this section we strangthen our assumption to $|\rho_1| > |\rho_2| > |\rho_j|$, $j > 2$. Then

$$\lim_{\lambda\to\infty} \frac{E(\lambda,t)}{(\rho_2/\rho_1)^\lambda} = d_2 \frac{(\rho_2-\rho_1)}{t-\rho_2} \quad . \tag{7.2}$$

The asymptotic error constant (Traub [14, p. 9]) is defined by

$$C(\lambda) = \lim_{t\to\rho_1} E(\lambda,t) \quad .$$

We conclude

$$\lim_{\lambda\to\infty} \frac{E(\lambda,t)}{C(\lambda)} = \frac{\rho_1-\rho_2}{t-\rho_2} \quad . \tag{7.3}$$

This result explains why the initial error ratio in the example of Section 3 is so small. For that example, $\rho_1 = -3$, $\rho_2 = 2$, $t = 100000$ and the initial ratio should be smaller than the asymptotic ratio by about -5×10^{-5} . This is indeed the case in the example.

If $B = P'$ we can draw an additional conclusion from (7.2).

In this case $d_2 = 1$. Let $P(t)$ and $Q(t)$ be two polynomials with the same dominant zeros ρ_1 and ρ_2 . We calculate two approximating sequences for ρ_1 , both starting at t_o but with one sequence calculated from P and the other from Q . On a computer, for λ sufficiently large, the two sequences are essentially identical. To put it another way, the sequence of approximants depends only on the two dominant zeros of P and is essentially independent of the remaining zeros.

8. TWO VARIATIONS OF THE BASIC ALGORITHM

In the following two variations the same sequence of approximants t_i, except for roundoff, is calculated as in the basic method described in Section 2. However the way in which the t_i are obtained is different.

Both variations are based on the following analysis. In Section 6 we showed that

$$\frac{G(0,t)}{P(t)} = \frac{B(t)}{P(t)} = \sum_{k=0}^{\infty} \frac{\beta(k)}{t^{k+1}} \quad . \tag{8.1}$$

Let $B(t) = \sum_{i=0}^{n-1} b_i t^{n-1-i}$. By comparing coefficients in (8.1), we conclude that for $B(t)$ given, $\beta(0)$, $\beta(1), \ldots, \beta(n-1)$ are determined by

$$\sum_{r=0}^{j} a_r \beta(j-r) = b_j \ , \ j = 0,1,\ldots, n-1 \quad . \tag{8.2}$$

For $j \geq n$ the $\beta(j)$ satisfy

$$\sum_{r=0}^{n} a_r \beta(j-r) = 0 \quad . \tag{8.3}$$

We can now associate $\beta(0)$, $\beta(1)$, ..., $\beta(n-1)$ with $B(t)$ in either of two ways. We can choose either the set $\beta(0)$, $\beta(1), \ldots, \beta(n-1)$ or $B(t)$ arbitrarily and determine the other by (8.2). In either case $\beta(j)$, $j \geq n$, is calculated using (8.3). (We might add parenthetically that if $B = P'$, then (8.2) are Newton relations for the power sums $\beta(\lambda)$.)

We now turn to variation one. Define $\alpha_j(\lambda)$ by

$$G(\lambda,t) = \sum_{j=0}^{n-1} \alpha_j(\lambda) t^{n-1-j} \quad .$$

It follows from (6.3) that

$$\alpha_j(\lambda) = \sum_{r=0}^{j} a_{j-r} \beta(\lambda+r) \quad . \tag{8.4}$$

This variation may now be described as follows. Compute the $\beta(j)$ up to $\beta(\lambda+n-1)$ using (8.2) and (8.3) and compute $\alpha_j(\lambda)$ using (8.4). This gives an explicit formula for $G(\lambda,t)$ and hence for $\varphi(\lambda,t)$.

Observe that this variation consists of a Bernoulli calculation followed by iteration.

The second variation is based on the fact that in the iteration

$$t_{i+1} = \varphi(\lambda,t_i)$$

only the numbers $G(\lambda,t_i)$, not $G(\lambda,t)$ itself, are required. We form the $\beta(j)$ up to $\beta(\lambda-1)$ using (8.2) and (8.3). Then form the sequence of numbers

$$(8.5) \qquad G(j+1,t_o) = t_o G(j,t_o) - \beta(j)P(t_o)\ ,\ j = 0,1,\ldots,\lambda-1\ ,$$

and use $G(\lambda,t_o)$ to calculate t_1. Then use (8.5) with t_o replaced by t_1, and so on.

9. AN ITERATION FUNCTION FOR THE SMALLEST ZERO

The iteration function $\varphi(\lambda,t)$ is used to calculate the largest zero of P. To calculate the smallest zero, we could calculate the largest zero of $t^nP(\frac{1}{t})$. We introduce a sequence of polynomials $H(\lambda,t)$ which may be used to construct iteration functions for the smallest zero directly.

It is convenient in this section to assume that $p(t)$, the polynomial whose smallest zero we seek to calculate, is normalized so that $p(0) = 1$. Let the zeros of $p(t)$ be $\alpha_1, \alpha_2, \ldots, \alpha_n$ with $|\alpha_1| < |\alpha_i|$, $i > 1$. Let $b(t)$ be an arbitrary polynomial of degree at most $n-1$ such that $b(\alpha_1) \neq 0$.

Define

$$H(0,t) = b(t)$$

$$H(\lambda+1,t) = \frac{H(\lambda,t)-\delta_o(\lambda)p(t)}{t} \tag{9.1}$$

where

$$\delta_o(\lambda) = H(\lambda,0) \quad .$$

An approximating sequence is defined by

$$t_{i+1} = \Phi(\lambda,t_i) \tag{9.2}$$

where

$$\Phi(\lambda,t) = \frac{t}{1-\frac{p(t)}{\widetilde{H}(\lambda,t)}} \tag{9.3}$$

with

$$\widetilde{H}(\lambda,t) = H(\lambda,t)/\delta_o(\lambda) \quad .$$

From (9.1), we also have

$$\Phi(\lambda,t) = \frac{H(\lambda,t)}{H(\lambda+1,t)} \quad . \tag{9.4}$$

10. PROPERTIES OF THE H POLYNOMIALS

From the defining recusion for the H polynomials,

$$H(0,t) = b(t)$$

$$H(\lambda+1,t) = \frac{H(\lambda,t)-\delta_o(\lambda)p(t)}{t} \quad , \tag{10.1}$$

we obtain the representation

$$H(\lambda,t) = \sum_{i=1}^{n} q_i \alpha_i^{-\lambda} \frac{p(t)}{t-\alpha_i} \quad , \quad q_i = \frac{b(\alpha_i)}{p'(\alpha_i)} \quad . \tag{10.2}$$

It follows that

$$\delta_o(\lambda) = -\sum_{i=1}^{n} q_i \alpha_i^{-\lambda-1} \tag{10.3}$$

and hence that $\delta_o(\lambda)$ does not vanish for λ sufficiently large. From (10.2) and (10.3) we conclude that

$$\lim_{\lambda\to\infty} \widetilde{H}(\lambda,t) = \lim_{\lambda\to\infty} \frac{H(\lambda,t)}{\delta_o(\lambda)} = \frac{p(t)}{1-\frac{t}{\alpha_1}} \tag{10.4}$$

for all finite t.

The H polynomials possess a property which is analogous to a G polynomial property discussed in Section 6. We expand $H(\lambda,t)/p(t)$ into a Taylor series around the origin. Let

$$\frac{H(\lambda,t)}{p(t)} = \sum_{k=0}^{\infty} e_k(\lambda) t^k \tag{10.5}$$

Let

$$\gamma(\lambda) = \sum_{i=1}^{n} q_i \alpha_i^{-\lambda} .$$

Clearly, $e_o(\lambda) = \delta_o(\lambda) = -\gamma(\lambda+1)$. Write the recurrence for $H(\lambda,t)$ as

$$\frac{H(\lambda+1,t)}{p(t)} = \frac{1}{t}\left[\frac{H(\lambda,t)}{p(t)} - \delta_o(\lambda)\right] \tag{10.6}$$

Then we conclude that

$$e_{k+1}(\lambda) = e_k(\lambda+1) \tag{10.7}$$

Hence the right side of (10.6) may be viewed as the operation of performing a left shift upon the vector of coefficients of the Taylor series.

From (10.7)

$$e_k(\lambda) = e_o(\lambda+k) = -\gamma(\lambda+k+1) \quad .$$

Hence

$$\frac{H(\lambda,t)}{p(t)} = t^{-\lambda}\left[\frac{b(t)}{p(t)} + \sum_{k=0}^{\lambda-1}\gamma(k+1)t^k\right] \quad .$$

Thus, except for a factor of $t^{-\lambda}$, $H(\lambda,t)/p(t)$ is just the remainder of the series for $H(0,t)/p(t)$ after λ terms.

11. CALCULATION OF MULTIPLE ZEROS

Until now we have restricted ourselves to polynomials all of whose zeros are simple. We turn to the case where the polynomial has multiple zeros. There are no essential difficulties. If the dominant zero is multiple, $P(t)$ can only be evaluated to a certain accuracy but this is common to all iterative methods which require the evaluation of $P(t)$.

We first prove a fundamental

THEOREM. Let P have n distinct zeros ρ_i where the multiplicity of ρ_i is m_i . Then for all λ

$$\frac{G(\lambda,t)}{P(t)} = \sum_{i=1}^{n} \frac{m_i\rho_i^{\lambda}}{t-\rho_i} \quad . \tag{11.1}$$

PROOF. We proceed by induction on λ . If $\lambda = 0$, the result is well known. Assuming it holds for λ and substituting (11.1) into the recursion formula for the G polynomials yields the result immediately.

Observe that (11.1) implies that for all λ , $G(\lambda,t)$ has zeros of multiplicity $m_i - 1$ at ρ_i . Furthermore,

$$\lim_{\lambda\to\infty} \overline{G}(\lambda,t) = \frac{P(t)}{t-\rho_1} \quad .$$

Hence, for λ sufficiently large, the remaining n-1 zeros of $\overline{G}(\lambda,t)$ lie arbitrarily close to the subdominant zeros of P. Thus the iteration function will have no poles in the neighborhood of ρ_1.

Observe that the theorem is based on the choice $B(t) = P'(t)$. This shows that the restriction $B(\rho_1) \neq 0$ is not the appropriate condition in the case of a multiple zero. The reason for this is apparent if one compares the partial fraction expansion of $G(\lambda,t)/P(t)$ in the simple and multiple zero cases.

A detailed analysis of the multiple zero case will appear elsewhere.

12. CALCULATION OF COMPLEX CONJUGATE ZEROS

So far we have dealt with polynomials which have a zero of largest modulus or a zero of smallest modulus. We turn to the case of equimodular dominant zeros. Fortunately in the case of polynomial zeros it is sufficient to consider the case of either one zero of largest modulus or of a pair of complex conjugate zeros of largest modulus for the following reason.

A translation in the t plane replaces zeros of equal modulus by zeros of unequal modulus. In the case of a polynomial with real coefficients, a real translation will remove all zeros of equal modulus except for a pair of complex conjugate zeros. Hence only the two cases mentioned need be considered.

A discussion of how to effect the translation so as not to damage the zeros of P will appear elsewhere.

We turn to the calculation of a pair of complex conjugate zeros. In [17] we recently announced a theorem on global convergence of an iterative method for calculating complex zeros. In this section we describe one

method for calculating complex zeros and state the theorem of global convergence. Variations on and extensions of this method as well as proofs of our results will be published in a forthcoming paper.

The theory holds no matter what the relation between ρ_1 and ρ_2 requiring only $|\rho_1| > |\rho_i|$ and $|\rho_2| > |\rho_i|$, $i > 2$. Here we restrict ourselves to ρ_1 and ρ_2 complex conjugate.

If $|\rho_1| = |\rho_2|$, then the normalized G polynomials do not converge. Let

$$I(\lambda,t) = \beta(\lambda)G(\lambda+1,t) - \beta(\lambda+1)G(\lambda,t)$$

$$J(\lambda,t) = \beta(\lambda)G(\lambda+2,t) - \beta(\lambda+2)G(\lambda,t) \quad . \tag{12.1}$$

Then

$$\overline{I}(\lambda,t) \to \frac{P(t)}{(t-\rho_1)(t-\rho_2)} \quad ,$$

$$\overline{J}(\lambda,t) \to \frac{P(t)}{(t-\rho_1)(t-\rho_2)} \quad .$$

Recursions involving only the I and J polynomials and not depending on the G polynomials have been developed. These recursions may be of advantage in numerical calculations.

From the I and J polynomials an iteration function may be constructed as follows. We define a polynomial which is quadratic in u and has coefficients which are polynomials in t of degree at most n-2,

$$F_2(u,\lambda,t) = I(\lambda,t)u^2 - J(\lambda,t)u + I(\lambda+1,t) \quad .$$

Let λ be a fixed integer and let t_o be an arbitrary point in the extended complex plane not equal to a subdominant zero. Define an iteration by

$$F_2(t_{i+1}\ ,\ \lambda\ ,\ t_i) = 0 \quad .$$

It can be shown that for all t_i, and for λ sufficiently large, this quadratic has a zero in the upper half plane and a zero in the lower half plane. Chhose t_{i+1} as the zero in the upper half plane and define ψ by $t_{i+1} = \psi(\lambda, t_i)$. Label ρ_1 as the zero in the upper half plane. Then we have the following.

THEOREM. Let the zeros ρ_i of the polynomial P be distinct with ρ_1 and ρ_2 complex conjugate and $|\rho_1| > |\rho_i|$, $i > 2$. Let t_o be an arbitrary point in the extended complex plane such that $t_o \neq \rho_3, \ldots, \rho_n$ and let $t_{i+1} = \psi(\lambda, t_i)$. Then for all λ sufficiently large but fixed, the sequence t_i is defined for all i and $t_i \to \rho_1$.

13. A NUMERICAL EXAMPLE

For illustration of the method discribed in the previous section we calculate the dominant zero of

$$P(t) = t^4 - 4.2t^3 + 8.7125t^2 - 9.025t + 4.625 \quad .$$

Its zeros are

$$\rho_1 = 1.1 + 1.05i$$
$$\rho_2 = 1.1 - 1.05i$$
$$\rho_3 = 1 + i$$
$$\rho_4 = 1 - i \quad .$$

Note that the zeros are pairwise quite close together.

We choose $B(t) = P'(t)$, $\lambda = 96$, and choose our initial approximation as $t_o = 1000$. We obtain the sequence of approximations exhibited in Table 2. In the right hand column we exhibit the ratios of the moduli of the errors. As in the example of Section 3 we observe

TABLE 2. SEQUENCE OF APPROXIMANTS

i	t_i	$\lvert t_{i+1}-\rho_1\rvert/\lvert t_i-\rho_1\rvert$
0	1000.	
1	1.10009 + 1.04997i	9.5×10^{-8}
2	1.10000003 + 1.04999992i	9.0×10^{-4}
3	1.09999999997 + 1.04999999992i	9.0×10^{-4}

$$\rho_1 = 1.1 + 1.05i$$

that all the ratios are small and that the initial ratio is particularly small. Again this can be quantitatively explained.

14. CALCULATION OF ZEROS OF ANALYTIC FUNCTIONS

Let

$$f(t) = \sum_{j=0}^{\infty} a_j t^j \quad , \quad a_o = 1$$

be a power series which converges in a circle about the origin. Suppose that $f(t)$ has a zero of smallest magnitude. Then we can define analytic functions $H(\lambda,t)$ by the recursion of (9.1). Results analogous to those in the polynomial case can be developed here.

Since we cannot actually form the analytic functions $H(\lambda,t)$, we cannot use the basic method. There are a number of other possibilities and we merely sketch two of them.

The first takes a section of the power series of degree n and uses it instead of f itself. A section of degree 1000 would offer no difficulties. The size of λ which is needed to separate out the effect of the dominant zero depends on the ratio of dominant to subdominant zero and not on the degree of the section one takes. Hence quite a modest choice of λ, much smaller than the degree of the section, should be sufficient. Since $G(\lambda,t)$ can be formed in λn multiplications and since each iteration takes about $2n$ multiplications, the process is reasonably economical even for large values of n.

A second possibility is to use the second variation of the basic method as described in Section 8. The variation is used with the H recursion rather than the G recursion. The constants appearing in the H recursion can be precomputed by an appropriate generalization of (8.2) and (8.3) which amounts to calculating the coefficients of the

Taylor series for $H(0,t)/f(t)$. This last mentioned process is just the computation required for the application of König's method [10].

15. COMPUTER IMPLEMENTATION

In the computer implementation of the type of methods described here, the program should decide automatically on the value of λ at which to start iteration, and as to whether or not there is a zero of largest modulus. Such decisions should be made by monitoring the numbers produced during the calculation of the G polynomials. A number of strategies are available and will be discussed elsewhere.

16. BIBLIOGRAPHIC REMARKS

Schröder [12] in his classic 1870 paper introduced certain symmetric functions of zeros. These symmetric functions are just the derivatives of the rational functions $G(\lambda,t)/P(t)$. He derived a number of the properties of these functions. Since Schröder restricted himself to low values of λ for which explicit formulas could be obtained, he did not find globally convergent iteration functions.

In 1941, Sebastião e Silva [13] defined G polynomials as the remainder of the division of t^{λ} by $P(t)$ and gave a long proof that the normalized G polynomials converge to $P(t)/(t-\rho_1)$. His work has been continued by Aparo [1], [2].

G polynomials are used by Bauer [3], [4] in an important series of papers which appeared in the mid-1950's. H polynomials appear in a paper by Bauer and Samelson [5].

Sebastião e Silva, Aparo, and Bauer are concerned with quadratically convergent versions of Bernoulli-Jacobi-Aitken type methods for the factorization of polynomials. Thus they continue the first state of our two-stage process to the limit.

Underlying many of the methods for calculating zeros are theorems concerning the coefficients of a function which has poles on its circle of convergence. Papers by König [10] and Hadamard [7] are classic. A perceptive account is given by Householder [9, Chapter 3]. The method we have discussed here may be incorporated in this framework.

Our work has links with the QD algorithm (Rutishauser [11], Henrici [8]) which will be explored elsewhere.

Finally we note a different application of G polynomials. Traub [15] uses G polynomials with the variable t replaced by the translation operator E, to give a new derivation of the formula for the general solution of a linear inhomogeneous difference equation with constant coefficients.

Additional bibliographic references may be found in Traub [16] and in the papers by Bauer.

Acknowledgement

I would like to thank my colleague Professor W. Kahan of the University of Toronto and Stanford University for many stimulating conversations.

REFERENCES

1. APARO, E., Applicazione di Un Nuovo Metodo per la Risoluzione Numerica delle Equazioni Algebriche. Bol. Soc. Portuguesa Mat., Sér. A, 1 (1948), 49-57.

2. _____, Un Procedimento Iterativo per la Risoluzione Numerica delle Equazioni Algebriche. Ricerca Sci. 24 (1954), 1003-1005.

3. BAUER, F. L., Beiträge zur Entwicklung numerische Verfahren für programmgesteuerte Rechenanlagen. I. Quadratisch konvergente Durchführung der Bernoulli-Jacobischen Methode zur Nullstellenbestimmung von Polynomen. Bayer. Akad. Wiss. Math. -Nat. Kl. S.-B. (1954), 275-303. II. Direkte Faktorisierung eines Polynoms. Bayer. Akad. Wiss. Math. -Nat. Kl. S. -B. (1956), 163-203.

4. _____, Das Verfahren der abgekürtzen Iteration fur algebraische Eigenwertprobleme, inbesondere zur Nullstellenbestimmung eines Polynoms. Z. Angew. Math. Phys. 7 (1956), 17-32.

5. BAUER, F. L. and Samelson, K., Polynomkerne und Iterationsverfakren. Math. Zeitschr. 67 (1957), 93-98.

6. FORSYTHE, G. E., Singularity and Near Singularity in Numerical Analysis. Amer. Math. Mo. 65 (1958), 229-240.

7. HADAMARD, J., Essai sur l'étude des functions données par leur dévelopment de Taylor. J. Math. Pures Appl. (4) 8 (1892), 101-186.

8. HENRICI, P., The Quotient-Difference Algorithm. NBS Applied Mathematics Series 49 (1958), 23-46.

9. HOUSEHOLDER, A. S., Principles of Numerical Analysis, McGraw-Hill Book Company, Incorporated, New York, 1953.

References (Cont'd)

10. KÖNIG, J., Über eine Eigenschaft der Potenzreihen. Math. Ann. 23 (1884), 447-449.

11. RUTISHAUSER, H., Der Quotienten-Differenzen-Algorithmus. Mitteilung aus dem Institut fur Angew. Math., No. 7 Birkhauser Verlag, 1957.

12. SCHRÖDER, E. Über unendlich viele Algorithmen zur Auflösung der Gleichungen. Math. Ann. 2 (1870), 317-365.

13. SEBASTIAO e SILVA, J., Sur une méthode d'approximation semblable à celle de Gräffe. Portugal. Math. 2 (1941), 271-279.

14. TRAUB, J. F., Iterative Methods for the Solution of Equations, Prentice-Hall, Incorporated, Englewood Cliffs, New Jersey, 1964.

15. _____, Solution of Linear Difference and Differential Equations. Bull. Amer. Math. Soc. 71 (1965), 538-541.

16. _____, A Class of Globally Convergent Iteration Functions for the Solution of Polynomial Equations. Math. Comp. 20 (1966), 113-138.

17. _____, Proof of Global Convergence of an Iterative Method for Calculating Complex Zeros of a Polynomial. Notices Amer. Math. Soc. 13 (1966), 117.