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On the Complexity of Montonic Inheritance with Roles

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ABSTRACT

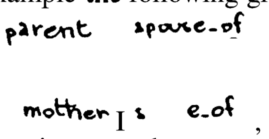
We investigate the complexity of reasoning with monotonic inheritance hierarchies that contain, beside **ISA edges, also ROLE** (or **FUNCTION) edges.** A **ROLE** edge is an edge **labelled** with a name such as **spouse-of** or **brother-of.** We call such networks **ISAR** *networks.* Given a network with **n** vertices and **m** edges, we consider two problems: **(P,)** determining whether the network implies an **isa** relation between two particular nodes, and **(P,)** determining ***all* isa** relations implied by the network. As is well known, without **ROLE edges** the time complexity' of **P,** is **O(m),** and the time complexity of P_2 is $O(n^3)$. Unfortunately, the results do not extend naturally to **ISAR** networks, except in a very restricted case. For general **ISAR** network we **first** give an polynomial algorithm by an easy reduction to proposional **Horn** theory. As the degree of the polynomial is quite high ($O(mn^4)$ for P,, $O(mn^6)$ for **P,),** we then develop a more direct algorithm. For both P, and P_2 its complexity is $O(n^3 + m^2)$. Actually, a finer analysis of the **algorithm** reveals a complexity of $O(nr(\log r) + n^2r + n^3)$, where **r** is the number of different **ROLE** labels. One **corolary** is that if we fix the number of **ROLE** labels, the complexity of our algorithm drops back to $O(n^3)$.

1. INTRODUCTION

Inheritance systems are a common framework for representing knowledge, in both AI and the database community. In these systems objects are organized hierarchically, and properties of objects are inherited by those below them in the hierarchy. For example, if it is recorded in this knowledge base that mothers are parents and that parents are responsible people, it may be concluded that mothers too are responsible.

As is well known, an inheritance system may be represented by a directed graph. The vertices in the graph are all of the same kind, and they each represent a class of objects. Arcs, on the other hand, come in several varieties, and there **has** been less uniformity among **the** various inheritance schemes in this respect. Beside the basic **ISA** type of arc, denoting class inclusion and common to all systems, other types that have been mentioned are **ROLEs** (or **FUNCTIONs), RELATIONs,** and **IDENTITYs.** In the past few years much attention has been paid to the issue of *cancellation* of inheritance, that is, to systems which allow an object to override some property that it would otherwise inherit from another object higher in the hierarchy. *These* systems have **been** *called nonmonotonic* (since the set of properties does not increase monotonically as one descends the hierarchy); in contrast, systems without cancellation have been called *monotonic.* Most recent research in inheritance systems has been concerned with the semantics of inheritance. In particular there have been several results relating cancellations to nonmonotonic **logics** (Etherington, **1987),** (Touretzky, **1986),** (Touretzky et al., 1987).

Our concern in this paper is different, as we look at the *complexity* of reasoning with inheritance networks. Consider a network with vertices V and edges **E,** and let $|V| = n$ and $|E| = m$. **As** is **well** known, if **all** the edges are **ISA** edges (such simple networks have been called *tuxonomic)* then in time **O(m)** one can determine whether the network implies an **ISA** relation between two particular nodes, and in time $O(nm)$ (and therefore in time $O(n^3)$) one can **find** all the implied **ISA** relations in the graph. If E contains other types of edge or if cancellation is allowed then the problem becomes harder. We know of relatively few results in this direction, including ones by Touretzky (1986) and **Borgida** (1989). Some relevant results involving negative and positive links are found in **Thomason** (1986). There are also results involving **RELATIONs** and **IDENTITYs** in **Thomason** (1989). We know of no results on the particular problem we consider, which is **to allow E to** contain **ROLEs** as well as **ISA** edges, and to prohibit cancellation; we call these ISAR networks. We preclude cancellation not because we consider it unimportant, but because we would like to understand the monotonic case **first.** As will be seen, it is by no means straightforward. The problem lies in a new closure rule that is provided by the interaction between ROLEs and ISA edges. Consider for example the following graph:



Intuitively, since mothers are parents, spouses of mothers are also spouses of parents. In other words, an ISA relation is implied about the two right nodes.

Of course, this intuitive claim needs to be formalized, and we will indeed do that. We will then consider the complexity of determining the implied ISA relations in such a network. In a very restricted type of ISAR networks we will be able to salvage the $O(m)$ and $O(nm)$ results from the simple taxonomic case. For general ISAR networks we will offer a slightly costlier $O(n^3 + m^2)$ algorithm to find all the implied ISA relations. Actually, a finer analysis of our general algorithm reveals a complexity of $O(nr(\log r) + n^2r + n^3)$, where r is the number of ROLE labels in the network (we distinguish ROLE labels which are distinct, like brother-of and spouse-of, and actual ROLE edges, in which ROLE labels may be repeated). Note that we have $r \leq m$, but we do *not* have $m \leq n^2$, since, unlike ISA edges, we may have multiple ROLE edges between two nodes (the spouses of mothers are exactly the joint-tax-payers of mothers). Among other things, this finer analysis takes us back to $O(n^3)$ for an r bounded by a constant. As this is close to the best known algorithm for simple taxonomic networks it seems unlikely that this result can be significantly improved.

The remainder of the article is organized as follows. In section 2, we briefly define the semantics of ISAs and ROLEs, and based on these we provide provably complete conditions for determining all the implicit ISAs entailed by a given ISAR network. In section 3, we formally define the graph theoretic problem. In section 4, we briefly recall the results on taxonomic hierarchies, all well known. In section 5, we finally turn to the complexity of reasoning with ISAR networks. In section 5.1, we extend the results of section 4 to a restricted kind of ISAR networks which we call "equi-multiple inheritance"-ISAR (EMI-ISAR) networks. We then turn to the general case. First, in section 5.2, we provide a polynomial algorithm which reduces the problem to that of determining entailment by a propositional Horn theory. The degree of the polynomial turns out to be quite high, and so, in section 5.3, we give another, more direct algorithm, whose complexity was discussed above. Finally, in section 6 we summarize our results, compare them to previous results of which we are aware, and point to some open questions.

2. THE SYNTAX AND SEMANTICS OF ISAR NETWORKS

In order to be able to define our problem we first present the syntax and semantics for monotonic ISAR networks. Their syntax is defined as follows.

Definition 1: Let V and L be two disjoint sets. An *ISAR network* is a triple $< V, E_I, E_R >$ where $E_I \subseteq V \times V$, $E_R \subseteq V \times V \times L$ and it satisfies:

1) If $(a,b,p) \in E_R$ and $(a,c,p) \in E_R$ then $b = c$;
2) If $p \in L$ then there are $a \in V$ and $b \in V$ such that $(a,b,p) \in E_R$.

V is the set of *vertices*, L is the set of ROLE *labels*, E_I is the set of *ISA edges* and E_R is the set of ROLE *edges*.

The second condition in the above definition is not essential, but it guarantees that any ROLE label indeed labels at least one ROLE edge, which is convenient. We now define their semantics.

Definition 2: Let $N = < V, E_I, E_R >$ be an ISAR network and L the set of ROLE labels of N A *model* for N is a pair $< D, \psi >$ where D is a set and ψ is a (total) function on $V \cup L$ such that:

1) If $a \in V$ then $\psi(a) \subseteq D$;
2) If $p \in L$ then $\psi(p)$ is a partial function from D to D;
3) if $(a,b) \in E_I$ then $\psi(a) \subseteq \psi(b)$;
4) if $(a,b,p) \in E_R$ then $\psi(b) = \psi(p)(\psi(a))$.

Next we define two isa relations, one semantic and one syntatic.

Definition 3: Let $N = < V, E_I, E_R >$ be an ISAR network. The binary relation isa_1 on V is defined by: $isa_1(a,b)$ iff for every model $< D, \psi >$ for N, it is the case that $\psi(a) \subseteq \psi(b)$. We will denote the fact that $isa_1(a,b)$ holds by $N \models isa(a,b)$.

Definition 4: Let $N = < V, E_I, E_R >$ be an ISAR network. The binary relation isa_2 on V is the smallest set satisfying:

1) If $(a,b) \in E_I$ or $a = b$ then $(a,b) \in isa_2$;
2) (Rule1) If $(a,b) \in isa_2$ and $(b,c) \in isa_2$ then $(a,c) \in isa_2$;
3) (Rule2) If $(a,b) \in isa_2$, $(a,c,p) \in E_R$ and $(b,d,p) \in E_R$ then $(c,d) \in isa_2$.

We will denote the fact that $isa_2(a,b)$ holds by

$N \vdash isa(a,b)$.

The next theorem establishes that Isa, and isa_2 are actually the same relation.

Theorem I: (Soundneas and Completeness) Let $N = < V,E_I,E_R >$ be an ISAR network. For every $a \in V$ and $b \in V$, $N \vdash isa(a,b)$ iff $N \models isa(a,b)$.

Proof ($\rightarrow$)Note that if $(a,b) \in E_I$ then $N \models isa(a,b)$; and also that Rule1 and Rule2 are sound with respect to our semantics. ($\leftarrow$)We omit this part of the proof; it will be included in the long version of this paper.

Note that if E_R is empty then the ISAR network reduces to a simple taxonomic inheritance network.

3. FORMAL PROBLEM DEFINITION

Given the syntax and semantics of ISAR networks, we now formally define the two problems we will be addressing.

P_1. Input: an ISAR network $N = < V,E_I,E_R >$ and a pair of vertices x,y in V
Output: 'yes' if $N \models isa(x,y)$, 'no' otherwise

P_2. Input: an ISAR network $N = < V,E_I,E_R >$
Output: an ISAR network $N = < V,E_I',E_R >$ such that $E_I' = \{(x,y): N \models isa(x,y)\}$

If $|V| = n$ and $COMP_i$ is the time complexity of P_i $(i = 1,2)$, then clearly we have $COMP_2 \leq n^2 COMP_1$, since we solve P_2 by solving P_1 for each pair of nodes.

In the rest of this paper, the number of vertices, $|V|$, will be n, the number of edges, $|E_I| + |E_R|$, will be m and the number of ROLE labels, $|R|$, will be r. Note that $r \leq m$ and $m \leq rn^2$.

4. SIMPLE TAXONOMIC HIERARCHIES: A REVIEW

In this section we briefly review the well-known results for the case in which the network contains only ISA edges.

Theorem 2: There exists an O(m) algorithm for P_1.
Proof. Use, e.g., the depth-first search (DFS) algorithm for directed graphs (Aho et al., 1974).

in fact, DFS may be used to find in O(m) time *all* the nodes reachable from a given node. We therefore have the following:

Corollary 1: There exists an $O(nm)$ algorithm for P_2.
Proof. Run a DFS from each node.

There is also the well known direct algorithm for P_2:

Theorem 3: **There exists** an $O(n^3)$ **algorithm** for P_2.
Proof. Use the dynamic programming algorithm of, e.g., (Aho et al., 1974).

In fact, there exists a theoretically even better algorithm for P_2, whose complexity is about $O(n^{2.7})$. However, this theoretical result has not been translated to a practical advantage.

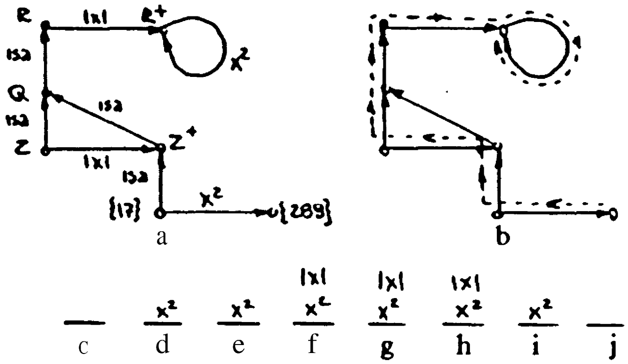
We mention these results for two reasons. First, as these are the best known results for taxonomic networks (and of course the linear result for P_1 is provably optimal) they form a lower bound for what we might expect for ISAR networks, and are good reference points against which to test our results. Second, the details of the algorithms mentioned above provide good insight into the qualitative increase in difficulty of ISAR networks. In the next section we discuss the DFS algorithm, and why it can be extended only to a limited class of ISAR networks. The dynamic programming algorithm, on the other hand, does not extend at all as far as we can see. Briefly, it relies on the property that if a path is decomposed at any vertex then each component is itself a path; that is true for simple taxonomic hierarchies, but not for general ISAR networks.

5. ALGORITHMS FOR ISAR NETWORKS

We now address the two problems defmed in section 3, P_1 and P_2, in the context of general ISAR networks. We start with a very efficient algorithm for a restricted class of ISAR networks. We then give an easy algorithm for the general case whose complexity, though polynomial, is uncomfortably high. Finally, we give a low polynomial algorithm for the general case.

5.1 EMI-ISAR networks

The DFS algorithm for taxonomic hierarchies extends paths into the graph, backtracks chronologically when a path is blocked, and never traverses the same edge twice. In this section we extend the algorithm to ISAR networks, introducing two major modifications. First, paths are extended in a way that is more complicated than simply following ISA edges. Second, in order to guarantee that we do not lose completeness by not traversing edges more than once (which guarantees linearity) we will need to impose a strong restriction on the network. Given the space limitations on this paper, we will only illustrate the algorithm through an example. Consider the simple network in Figure 1a.



Z:integers, Z^+**:nonnegative Z,**
Q:rationals, **R:reals,** R^+**:nonnegative R,**
|x|: the absolute value function, x^2**: squaring function**

Figure 1

Now consider the query '**isa({289},R⁺)**'. This query should succeed due to the path shown in Figure lb, which consists of three **types** of edge traversal: going back on **ROLE** edges (e.g., Z^+ to Z), going up **ISA edges (e.g., Z to Q),** and going forward **on ROLE** edges (e.g., **R to** R^+**).** We will call these respectively left, up and right traversal. Left and up **traversals** have no preconditions. Right **traversal** has a precondition that it not immediately follow a left **traversal,** and that the last left traversal to precede it **was** along a **ROLE** with the same label. To implement this we maintain a stack as we develop a path: up traversal does not **affect** the stack, back traversal pushes the **ROLE label** onto the stack, right traversal pops the stack (and has the precondition mentioned above). Figures 1c-1j illustrate the stack at all the vertices along the path in Figure lb.

Lemma 1: Let **N** be an **ISAR** network. Then **N** $\models$ isa(x,y) **iff** there is a path of the sort described above that starts at x with an empty stack and ends at y with an empty stack.

The only question that remains is how to determine efficiently whether such a path exists. Unfortunately, in **ISAR** networks with multiple inheritance we will in general need to traverse some edges many times. A simple example exists already in Figure la: if the first path developed is **{289}{ 17}**Z^+**QR,** then at that point backtracking must occur. If we are not allowed to traverse the edge **QR** twice, then we will not discover the path **{289}{17}**Z^+**ZQRR**$^+$**R**$^+$, and thus miss a solution. In special case, however, it is safe to not traverse an edge twice:

Definition 5: The *label* of a path is the sequence of **ROLE** labels appearing in it, ignoring all **ISA** edges.

Definition 6: **An ISAR network is** an ***equi-multiple inheritance-ISAR*** network **(EMI-ISAR network) if for** any **two nodes** x and y, **all** path8 from x to y **have the same label.**

Theorem 4: In **the case** of **EMI-ISAR** networks there exists **an O(m)** algorithm for P,.

Proof. Develop paths of the sort described above in a depth-first fashion, backtracking chronologically, **never** traversing an edge twice.

In fact, just as in the simple taxonomic **case,** this **extended DFS** can be used to discover ***all*** nodes to which a path exists from **a** given node. We thus get the following:

Corollary 2: In the **case** of **EMI-ISAR** networks there exists an $O($ **nm) algorithm** for P_2.

Note that our results hold also when the network contains cycles.

5.2 Reducing general ISAR networks to propositional Horn theory

We **now start** to look at the general **case** of **ISAR** networks. In this section we pursue an easy way out, namely to reduce the graph theoretic problem to the problem of deciding a query about a propositional Horn theory, which is known to be decidable in linear time (Dowling and **Gallier,** 1984). Unfortunately, the resulting **datalog** theory will not be linear in the size of the **ISAR** network.

Let N = < V,E$_I$,E$_R$> be an ISAR network. We construct a Horn theory **Th(N)** as follows. First, for each three vertices x,y,z in **N,** we construct a clause

$$\text{isa}(x,y) < - \text{isa}(x,z) \wedge \text{isa}(z,y)$$

Then, for each four vertices v,x,y,z in **N** and each **ROLE** label l we construct a clause

$$\text{isa}(x,y) < - \text{isa}(v,z) \wedge \text{role}(l,v,x) \wedge \text{role}(l,z,y)$$

Finally, for every pair **(a,b)** in **E,** we add a predicate **isa(a,b),** and for every triple **(a,b,p)** in E_R we add a predicate **role(p,a,b)**.

Theorem 5: There exists an $O(rn^4)$ (and thus $O(mn^4)$) **algorithm** for **P,.**

Proof. From Theorem 1 we have that **N** $\models$ isa(x,y) iff **Th(N)** $\models$ isa(x,y). The latter can be decided in time linear in **Th(N).** The number of clauses in **Th(N)** is $O(n^3 + rn^4) = O(rn^4)$.

Corollary 3: There exists an $O(rn^6)$ (and thus $O(mn^6)$) algorithm for **P,.**

5.3 An efficient algorithm for general ISAR networks

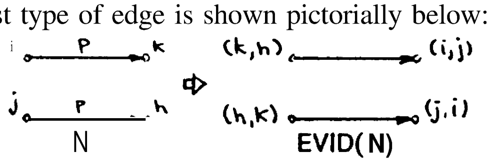
The degree of the polynomial in the previous algorithm is a bit too high for comfort. We now offer a more direct algorithm **whose complexity is much** lower.

Definition 7: A directed AND/OR *graph* **is one** in which the set of edges emanating **from** each node is partitioned into sets, each set called an **AND-set** of that node (single **edges** arc viewed as singleton sets). A *path* in such **a** graph **is a** rooted tree such that the set of edges in the tree emanating from each **vertex** forms an **AND-set** of that **vertex in** the **AND/OR** graph. Searching an **AND/OR** graph from a given vertex means starting with a path consisting of the node itself, and iteratively extending it.

Definition 8: **Let N = <V,E$_I$,E$_R$ > be** an **ISAR** network. The *evidence graph of* N is the **AND/OR** directed graph **EVID(N) = < V^2,E'>** where

E'= {((k,l),(i,j)): for some **p, (i,k,p)** and (j,l,p) are both in E_R}
U { {((i,k),(i,j)),((i,k),(j,k))}: i,j,k in V}.

The first type of edge is shown pictorially below:



The intuition behind the construction is the following: an **AND-set** of a vertex **(i,j)** in the evidence graph is evidence that **(i,j)** is in the **isa** relation. More precisely, we have the following:

Definition 9: **Let N = < V,E$_I$,E$_R$ > be an ISAR** network. A path rooted at **(a,b) in EVID(N)** is said to be *grounded* if **a = b** or for all terminal nodes **(k,l)** in that path it is the case that **(k,l)** is **in E$_I$.**

Lemma 2: Let **N** = < V,E$_I$,E$_R$ > be an **ISAR** network and **i,j in** V. Then **N⊨isa(i,j) iff** there is a grounded path **in EVID(N)** rooted at **(i,j).**
Proof. (outline) By theorem 1, **N⊨isa(i,j)** if and **only if N⊢isa(i,j)**. By induction on the number of applications of Rule 1 and Rule2 (Defmition 4) we have that if N⊢isa(i,j). then there is a grounded path rooted at **(i,j) in** EVID(N). By induction on the the size of the path we may prove that if there is a grounded path rooted at **(i,j) in** EVID(N) then N⊢isa(i,j).

Lemma 3: It can be determined in time $O(m')$ simultaneously for **all** vertices **in EVID(N) whether there is a** grounded path rooted at them, where **m'** is the number of **edges in EVID(N).**
Proof. (outline) Conduct a **breadth-first** search **(BFS)** starting **from** all nodes **(i,j) such** that **(i,j)** is in **E$_I$**, moving *backwards* on edges, and extend a path beyond a vertex only when at least one of its **AND-sets** has **all** its members originate in **previously-**reached nodes.

The last lemma points to the reason for constructing the evidence graph. We now note that **m'** is bounded by the complexity of generating **EVID(N).** To complete the story, then, it remains to estimate this complexity. We **first** show an easy bound, and then look **more** closely at the algorithm to improve the complexity.

Theorem 6: There exists an $O(n^3 + m^2)$ algorithm for P_2.
Proof. The construction of the **edges in EVID(N)** that are due to the transitive closure is done in time $O(n^3)$. To construct the other edges, we look at all **pairs of ROLE edges (i,j)** and **(k,l),** and, **if** their **ROLE** labels agree, add **to EVID(N)** the edges ((i,k),(j,l)) and ((j,l),(i,k)). The total number of edge-pairs is $O(m^2)$. Thus the total complexity of the algorithm is $O(n^3 + m^2)$.

Recall that **in ISAR** networks there is no necessary relation between the number of vertices and the number of edges. However, if it happens that $m = O(n^2)$, we have that the algorithm is of complexity $O(n^4)$. We now improve on this by a more careful construction of the evidence graph.

Theorem 7: There exists an $O(nr(\log r) + n^2 r + n^3)$ algorithm for P_2, where r is the number of different **ROLE** labels.
Proof. We create the **first n^3** edges as before. Then, rather than blindly compare all pairs of edges, we do the following.

1) Create a **list** for each vertex of **all** the **ROLE** edges emanating from it and their associated label. A typical **list** will have the form **i: $(l_2,i_1),(l_5,i_3)$, ...** (where **i, i,** and i_3 **are** vertices, and **l,** and **l, are ROLE** labels);
2) Sort each of these lists by the label component;
3) For each pair of vertices **i,j,** scan their lists in parallel to see which role labels they share. If you encounter the pair **(p,k)** in **i's** list and the pair (p,l) in j's list, add the edges ((k,l),(i,j)) and ((l,k),(j,i)).

Complexity of the steps:

1) $O(m)$;
2) $O(nr(log\,r))$ (note that each list is of length r at most);
3) $O(n^2r)$ (scanning the sorted lists is linear in their length, r, and there are n^2 pairs of vertices).

We also note that we have $m \leq n^2r$, and so the total complexity of creating the evidence graph is $O(nr(log\,r) + n^2r + n^3)$.

Corollary 4: If the number of **ROLE** labels is bounded by a constant, there is an $O(n^3)$ algorithm for P_2.

We note that as this is realistically the lowest complexity known for transitive closure, we hould not hope to improve on this.

6. SUMMARY AND DISCUSSION

We have offered new results on the complexity of reasoning with inheritance hierarchies with **ROLEs**, or **ISAR** networks. We defined two problems, P_1 (determining whether a **ISAR** network implies an **isa** relation on two nodes) and P_2 (finding the closure of the **isa** relation). Let n be the number of vertices of an **ISAR** network, m the number of edges, and r the number of distinct **ROLE** labels. To somewhat crudely summarize our results, we have the following.

	no ROLEs	EMI-ISAR	Horn alg	direct alg	fixed labels
P_1	n	n	mn^4	m^2+n^3	n^3
P_2	nm	nm	mn^6	m^2+n^3	n^3

The only results bearing directly on **ISAR** networks with which we are familiar are due to Borgida (1989). His results include NP-Hardness for networks with cancellation, and polynomial results for two other problems. We do not yet understand well the relation between his results and ours. There appear to be few other complexity results. We are aware of Touretzky's (1986) polynomial algorithm for parallel networks with **RELATIONs**, but do not see an interaction with our work.

Our results leave open some interesting questions. Our general result for P_2 is somewhat worse than the $O(nm)$ of transitive closure; can it be improved? Another striking feature of our result is that in the general case we have identical results for P_1 and P_2, although at first glance it seems that P_1 is much easier. Actually, our experience with the problem leads us to conjecture that P_1 is not any easier, but it would be nice to have a result on that. Then there is a question about other ways to salvage the $O(n)$,$O(nm)$ results from the simple taxonomic case: do there exist interesting classes of networks which permit that other than **EMI-ISAR** networks? Finally, what happens when we add other features to the network, such as **RELATIONs** or cancellation? We conjecture that at least in the latter case the problem in general becomes intractable, which seems to agree with Borgida's result mentioned above.

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