THE NUMBER OF SDR'S IN CERTAIN REGULAR SYSTEMS

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Abstract

Let $(a_1,\ldots,a_k) = \bar{a}$ denote a vector of numbers, and let $C(\bar{a},n)$ denote the $n \times n$ cyclic matrix having $(a_1,\ldots,a_k,0,\ldots,0)$ as its first row. It is shown that the sequences $(\det C(\bar{a},n): n = k,k+1,\ldots)$ and $(\operatorname{per} C(\bar{a},n): n = k,k+1,\ldots)$ satisfy linear homogeneous difference equations with constant coefficients. The permanent, per C, of a matrix C is defined like the determanent except that one forgets about $(-1)^{\text{sign } \pi}$ where π is a permutation.

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Introduction

While she was a student at Lowell High School, Beverly Ross [2] generalized an exercise given by Marshall Hall Jr. [1], and found an elegant solution. Hall's exercise was posed in the context of systems of distinct representatives, or SDR's for short. Let $\bar{A} = (A_1, \ldots, A_m)$ denote an m-tuple of sets, then an m-tuple $(a_1, \ldots, a_m)$ with $a_i \in A_i$ for $i = 1, \ldots, m$ is an SDR of $\bar{A}$ if the elements $a_1, \ldots, a_m$ are all distinct. Hall's exercise is the case $m = 7$ of the following problem posed and solved by Ross: Let $A_i = \{i, i+1, i+2\}$ denote a 3-set of consecutive residue classes modulo m for $i = 1, \ldots, m$. The number of SDR's of $(A_i : i = 1, \ldots, m)$ is $2 + L_m$ where L_m is the m-th term of the Lucas sequence $1, 3, 4, 7, 11, \ldots$ defined by $L_1 = 1$, $L_2 = 3$ and $L_n = L_{n-1} + L_{n-2}$ for $n = 3, 4, \ldots$. For example, it follows from this result that the solution to Hall's exercise is $2 + L_7 = 31$.

In this note we give a new proof of Ross' theorem, and indicate a generalization.

Ross' Theorem

We shall require a simple result which appears in Ryser [3]; namely, the number of SDR's of an m-tuple $\bar{B} = (B_1, \ldots, B_m)$ of sets $B_1, \ldots, B_m$ is equal to the permanent of the incidence matrix of $\bar{B}$. Since this fact is an immediate consequence of definitions, we give them here. Let m and n denote natural numbers with $m \le n$, and let $B_1, \ldots, B_m$ denote subsets of $\{1, \ldots, n\}$. The incidence matrix $[b(i,j)]$ of $\bar{B} = (B_1, \ldots, B_m)$ is defined by

$$b(i,j) = \begin{cases} 1 \;, & \text{if } j \in B_i \;, \\ 0 \;, & \text{if } j \notin B_i \;, \end{cases}$$

for $i = 1,\ldots,m$ and $j = 1,\ldots,n$. The <u>permanent</u> of an $m \times n$ matrix $[r(i,j)]$ is defined to be

$$\text{per}[r(i,j)] = \sum_{\pi} r(1,\pi 1) r(2,\pi 2) \; . \; . \; . \; r(m,\pi m)$$

where the index of summation extends over all one-to-one mappings π sending $\{1,\ldots,m\}$ into $\{1,\ldots,n\}$.

The incidence matrix C_m of the m-tuple $\bar{A} = (A_1,\ldots,A_m)$ of sets $A_1,\ldots,A_m$ considered by Ross is an $m \times m$ cyclic matrix having as its first row $(1,1,1,0,\ldots,0)$; that is, the first row has its first three components equal to 1 and the rest of its components equal to 0 . For example, the incidence matrix for Hall's exercise is

$$C_7 = \begin{vmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} .$$

Ross' Theorem is equivalent to showing that per $C_m = 2+L_m$. To do this, we define three sequences of matrices:

$$D_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} , \quad D_4 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} , \quad D_5 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} , \ldots ;$$

$$E_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad E_5 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \ldots ;$$

$$F_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad F_4 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad F_5 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \ldots .$$

Let per $C_m = c_m$, per $D_m = d_m$, per $E_m = e_m$, and per $F_m = f_m$. We use the following properties of the permanent function. First, the permanent of a 0-1 matrix is equal to the sum of the permanents of the minors of the 1's in a row or in a column of the matrix. Second, the permanent of a matrix is unchanged by permuting the rows or by permuting the columns of the matrix. Third, the permanent of a matrix having a row or column of 0's is equal to 0 . Fourth, the permanent of a square matrix is equal to the permanent of the transpose of the matrix. Expanding per C_m in terms of the minors of the 1's in the first row of C_m, we find

(1) $\qquad c_m = 2d_{m-1} + e_{m-1} \qquad (m = 4,5,\ldots)$.

Expanding per D_m in terms of the minors of the 1's in the first column of D_m, we find

(2) $\qquad d_m = e_{m-1} + f_{m-1} \qquad (m = 4,5,\ldots)$.

It is easy to show that

(3) $$e_m = e_{m-1} + e_{m-2} \qquad (m = 4,5,\dots),$$

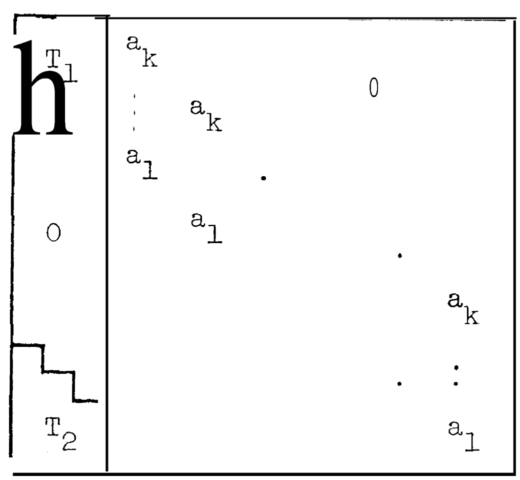
(4) $$f_m = f_{m\ 1} = \dots = f_3 = 1 .$$

Using the system (1) - (4) it is easy to show by induction that $e_m = F_{m+1}$, where F_m denotes the m-th term of the Fibonacci sequence $(1,1,2,3,\dots)$, $d_m = 1+F_m$, and $c_m = 2+2F_{m\ 1}+F_m = 2+F_{m\ 1}+F_{m+1} = 2+L_m$ for $m = 3,4,\dots$.

A Generalization

Let $\bar{a} = (a_1,\dots,a_k)$ denote a k-tuple of numbers and let T denote a $k \times (k-1)$ matrix having all of its entries in the set $\{0,a_1,\dots,a_k\}$. For each $n \geq k$ define an $n \times n$ matrix $C(T,n)$ as follows:

C(T,n) =



The first k-1 columns of C(T,n have the upper triangular half T_1 of T in the upper right corner, and the lower triangular half T_2 of T in the lower left corner. All other entries in the first k-1 columns of C(T,n) are 0 . The remaining n-k+1 columns of C(T,n) consist of n-k+1 cyclic shifts of the column $(a_k,\dots,a_2,a_1,0,\dots,0)$.

Given a $k \times (k-1)$ matrix T having all of its entries in $\{0, a_1, \ldots$ [illegible] and having $(t_1, \ldots, t_{k-1})$ as its top row, we expand per $C(T,n)$ by the minors of elements in the top row of $C(T,n)$. It turns out that these minors always have the form $C(T_i, n-1)$ where T_i is a $k \times (k-1)$ matrix having all its entries in $\{0, a_1, \ldots$ [illegible]. Thus, there exist $k \times (k-1)$ matrices T ,...,T having all their entries in $\{0, a_1, \ldots, a_k\}$ such that

$$(1) \qquad \text{per } C(T,n) = \sum_{i=1}^{k} t_i \text{ per } C(T_i, n-1)$$

where $t_k = a_k$. (If we are dealing with determanents, $(-1)^i$ must be put into the summand.)

We have an equation like (1) for each matrix T ; hence, we have a finite system of equations if we let T range over all possible $k \times (k-1)$ matrices with their entries in $\{0, a_1,$ [illegible]. The existence of this system of difference equations implies the existence of a difference equation satisfied by the sequence (per $C(T,n)$: $n = k, k+1, \ldots$) for each fixed matrix T . (This is also true for the sequence (det $C(T,n)$: $n = k, k+1, \ldots$).) A consequence of the foregoing is ths result proved by Ross, but evidently much more is true.

Let $r_1, \ldots, r_n$ denote natural numbers with $1 = r_1 < \ldots < r_n = k$, and for each natural number $m \geq k$ define the collection $\bar{A}_m = \{A_1, \ldots, A_m\}$ of sets A_i of residue classes modulo m where

$$A_i = \{r_1 + i, \ldots, r_n + i\} .$$

Let $a(m)$ denote the number of SDR's of $\bar{A}_m$, then the sequence $(a(m) : m = k, k+1, \ldots)$ satisfies a linear homogeneous difference equation with constant coefficients. The proof of this fact follows the proof of Ross' Theorem given in the preceding section.

Note that our existence theorem has a constructive proof, but we do not have an explicit expression for a difference equation satisfied by the sequence (per $\mathcal{C}(T,n): n = k,k+1,..)$. This gives rise to a host of interesting research problems. For example, give a difference equation satisfied by the sequence (per $C(k,n): n = k,k+1,...)$ where $C(k,n)$ is the cyclic $n \times n$ matrix having as its first row $(1,...,1,0,...,0)$ consisting of k 1's followed by $n-k$ 0's.

References

[1] Marshall Hall, Jr., Combinatorial Theory, Blaisdell Publishing Company, Waltham, Mass., 1967. (Problem 1, page 53.)

[2] Beverly Ross, "A Lucas Number Counting Problem," Fibonacci Quarterly, Vol. 10 (1972), pages 325-328.

[3] Herbert J. Ryser, "Combinatorial Mathematics," Number 14 of the Carus Mathematical Monographs, John Wiley, 1963.