Digital Computer Laboratory
Massachusetts Institute of Technology
Cambridge 39, Massachusetts

To: J. D. Porter

From: D. N. Arden and G. W. Patterson

Date: 17 June 1955

SUBJECT: WHIRLWIND DISPLAY PROGRAM

Vibrations in a Length of String

The oscilloscope display represents a length of string, held at both ends and then displaced from its rest position. Various input tapes can be used to set the string in different starting positions. The operating tape uses an equation of wave motion to simulate the vibrations. Manual inputs provide desired variations in:

a) The amplitude of vibration

b) The elasticity factor

c) Speed of display (number of points computed along the string)

1. Computer Operation

The program makes use of the following (decimal) registers:

32-115	Operating Tape 131-97-6
120-200	Input Tape 131-97-56
400-(400+2p)	(Stores computed values for string's present position; depends upon number of points computed along the string: p)

The manual inputs required by both the input and operating tapes are:

LMIR (222)	No. of points (p)	Suggested starting value: 0.00400
RMIR (223)	Magnitude of string displacement (m)	" " : 0.40000
FF #2	Scope amplitude adjustment	" " : 0.00010
FF #3	Elasticity factor	" " : 0.70000

Input and operating tapes may be read in simultaneously, if it is desired to make a continuous display of various conditions:

READ IN

Input Tape #1: 120 | 0r,si222 ....sp32

Start at 120

Operating Tape: 32 | cl,si4..................Start at cl

SET PROGRAM COUNTER to the first instruction of the input tape that has been chosen for use and start over.

RESTART COMPUTER. The vibration display will begin as soon as initial inputs have been computed. The display must be stopped manually.

Manual inputs for Elasticity and Amplitude may be varied at any time during operation. To vary the manual inputs for (p) and (m), the Program Counter must be reset to operate on the input tape again.

2. Theory

The reproduction of such a familiar physical motion as the vibrations in a piece of string is done by a continuous iteration, or the solution in real time, of a linear hyperbolic partial derivative function:

$$\frac{\partial^2 u}{\partial x^2} = 1/a^2 \ \frac{\partial^2 u}{\partial t^2}$$

u = lateral displacement of a point on the string

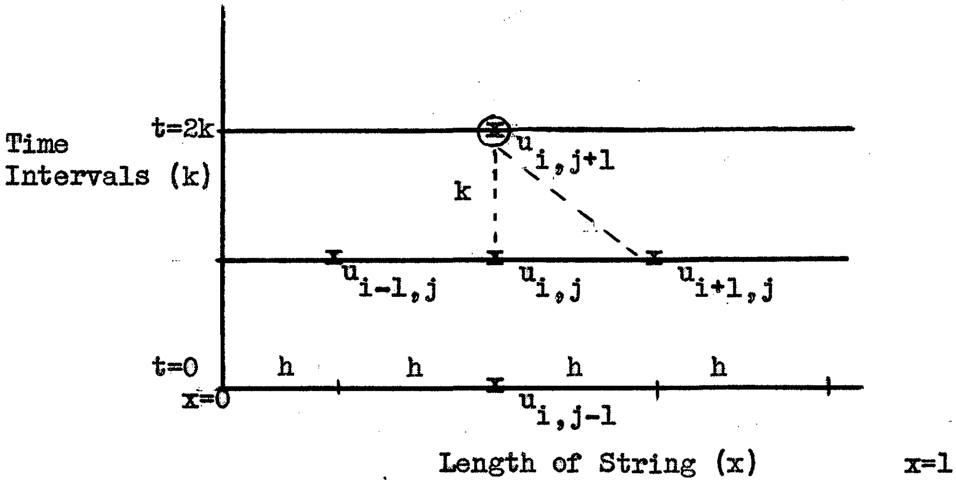
x = position of that point, as measured from the end of the string

a = elasticity of the string (or velocity of propagation of wave motion along the string)

To represent the position of the string, u is computed for each point on the string. The values of u then are displayed for each successive time interval (k).

Iteration

The step-by-step solution by digital computer is handled by using a linear equation that gives the displacement of one point ($u_{i,\ j+1}$) in terms of four adjacent known values of u (shown in the diagram):



From the diagram:

$$\frac{\partial^2 u}{\partial x^2} \cong \frac{u_{i+1,j} - \partial u_{i,j} + u_{i-1,j}}{h^2}$$

$$\frac{\partial^2 u}{\partial t^2} \cong \frac{u_{i,j+1} - \partial u_{i,j} + u_{i,j-1}}{k^2}$$

Substituting these approximations in the equation

$$\frac{\partial^2 u}{\partial^2 x} = 1/a^2 \; \frac{\partial^2 u}{\partial^2 t}$$

gives the result

$$u_{i,j+1} = k^2 a^2/h^2 \; (u_{i-1,j} - 2u_{i,j} + u_{i-1,j}) + 2u_{i,j} - u_{i,j-1} \quad .$$

Restriction: k^2a^2/h^2 must be greater than 1.

One requirement of this method is that the five points in the equation above fall between imaginary lines ("characteristics lines") drawn through the point $u_{i,j}$. The characteristics lines define the wavefront. That is, they pass through all points having the same displacement as $u_{i,j}$.

The slope of the two lines in known to be 1/a (i.e., $\partial t/\partial x = 1/a$). It can be seen from the diagram then that k/h must be greater than 1/a, or k^2a^2/h^2 must be greater than 1.

Procedure

Starting at the left end of the string (x=o), each point $u_{i,j+1}$ is computed, using the values which have been stored for the two previous time periods. When all the new values of u have been computed for the new time period (j+1), then the program steps to the next time period and begins again.

3. Performance

The program has worked well, using an input tape that in effect pulls the string offcenter like a bowstring.

It may be possible to revise the operating tape for more rapid iteration. However, the speed of the display must be kept low so that the persistence of the scope does not produce a blur of lines.

4. Future Use

This program was requested for demonstration purposes, as a standby for the Bouncing Ball display. Many variations are possible and may be added by the writing of additional input tapes.

An "impedance matching" display might be programmed, restraining one end of the string with variable restoring forces (springs). Determining the restoring force that gives no reflection at the end of the string is analogous to impedance matching on a transmission line

Signed: Dean N. Arden

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Digital Computer Laboratory
Massachusetts Institute of Technology
Cambridge 39, Massachusetts

To: S & EC Group

From: Richard Watson

Date: 20 September 1955

Subject: CHANGE IN VIBRATING STRING DISPLAY PROGRAM

Several different initial conditions and a variable real impedance at one end of the string have been added to the previously described program.

Replacing input tape 131-97-56 there are:

131-204-2 of the form 

131-204-6 of the form 

131-204-11 of the form 

131-204-12 gives a sine curve of as many degrees as points specified (contents of the LMIR)

The new operating tape is 131-204-10.

The equation for the right hand endpoint of the string is taken as

$$\beta Y_n + \frac{\gamma(Y_n - Y_{n-1})}{\Delta x} = 0 \qquad \text{or with } \gamma' = \frac{\gamma}{\Delta x}$$

$$\beta Y_n + \gamma' (Y_n - Y_{n-1}) = 0$$

FF #5 must contain γ' and FF #6 β.

If $\gamma' = 0$ we have the fixed end case. If $\beta = 0$ we have the free end case and if both $\neq 0$ we have the intermediate case.

It is assumed that $\beta = 1$ in calculations of the latter variable impedance case.

The manual inputs suggested in DCL-90 still hold except where sine curve input is used; here a value of 0.00550 octal (=360 decimal) seems better for the LMIR. Smaller amplitude (RMIR or FF #2) may seem desirable.

Signed: Richard E. Watson
Richard E. Watson gn

REW:gn