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On the Correlation Matrices of Trigonometric Product Functions

Abstract: Sets of waveforms called TPF's, which are products of trigonometric functions, may have application in comunications technology, network analysis and signal processing. This communication briefly reviews the characteristics of TPF's and presents methods for calculating their cross-correlation coefficients when 1) all TPF's in a set have harmonic factors with the same set of values, and 2) the harmonic factors have different values.

Introduction

This communication discusses a class of band-limited waveforms called *trigonometric product functions* (TPF's). These waveforms have potential application in communications technology, network analysis and signal processing. Of particular interest is the identification of sets of orthogonal TPF's, i.e., those TPF's whose cross-correlation coefficients are zero.

The orthogonal sets of TPF's have important advantages in comparison with other band-limited orthogonal signal sets used for data transmission.[1] With respect to frequency-divided codes[2-5] and block-orthogonal codes,[6,7] orthogonal TPF codes have few discontinuities both within the time interval over which they are defined, and at the ends of the interval. The time-divided type of orthogonal codes[2] has a significantly higher peak-to-average power ratio and requires considerably larger time-bandwidth products for a given size code.

Other publications by the author[8-10] contain more extensive discussions of the characteristics of TPF's than does this one. A technical report[11] gives some algorithms for finding orthogonal sets of TPF's. Additional information about orthogonal codes in general may be found in publications by Harmuth.[12-15] The main purpose of the present work is to show the methods that can be used to calculate the correlation coefficients for certain sets of TPF's.

Definitions

In terms of their application to communications technology, trigonometric product functions can be considered to be an encoding of a p-bit word into a product of sine and cosine factors. For example, the 3-bit words $000, \cdots, 111$ are represented by

$$
\begin{aligned}
000 &= \sin \alpha_1 x \sin \alpha_2 x \sin \alpha_3 x \\
001 &= \sin \alpha_1 x \sin \alpha_2 x \cos \alpha_3 x \\
&\vdots \\
111 &= \cos \alpha_1 x \cos \alpha_2 x \cos \alpha_3 x,
\end{aligned}
$$

where the sines represent ZEROS and the cosines represent ONES. The harmonic factors α_1, α_2, α_3 are distinct, positive integers assumed to be given in increasing order. A binary word, $b_1 \cdots b_p$, is represented in general by

$$b_1 \cdots b_p = \prod_{i=1}^{p} \sin (\alpha_i x + b_i \tfrac{1}{2}\pi), \tag{1}$$

where $x = \omega_0 t$; i.e., time is normalized with respect to a frequency appropriate to the operating band. The TPF's discussed in this paper are all truncated at $x = \pm\frac{1}{2}\pi$ to form pulses of duration $T = \pi/\omega_0$. We adopt the notation $F(p, k, q)$ to identify a given TPF (within a specified set, or *block*), which is assumed to be normalized to unit energy over its interval of definition. The value of p is the number of factors in the function and is called the *order* of the TPF; k is the integer value of the highest harmonic factor α_p and is called the *class* of the TPF; and q identifies the *rank* of the TPF in an ordered list of TPF's of given order and rank.[8] We call a set of TPF's a *block* if it contains all of the 2^p TPF's that each have the same harmonic factors. For a given order and class, there are $N_B = \binom{k-1}{k-p}$ blocks, and a total of $2^p N_B$ TPF's. The large number of TPF's that one may choose from to specify a set of signals with desired properties is ex-

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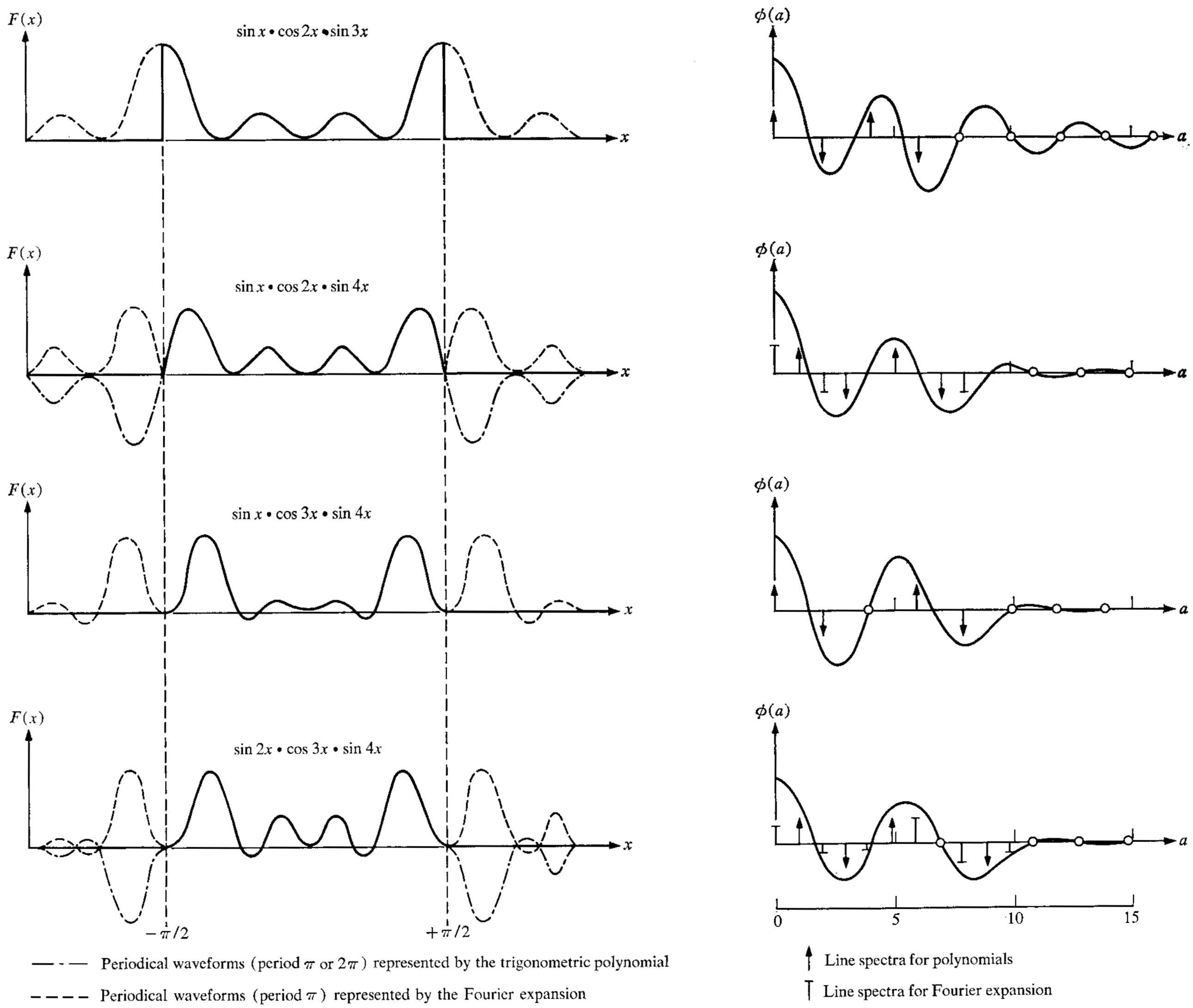

Figure 1 Some trigonometric product functions and their spectra.

emplified when one evaluates $2^p N_B$ for TPF's containing five or fewer harmonic factors. The calculation yields 270,262 different waveforms.

In their application to communications technology the product encodings convert binary data to analog forms in a manner similar to the function of data modems. By expanding the products into sums of harmonic terms, the encoding takes on the appearance of "multitone" or "parallel subcarrier" transmission of data. However, the product form provides a highly redundant encoding having resistance to channel noise and to interference from other users on a channel operating in a random-access mode.

Figure 1 shows a selection of TPF's and their spectra.

The main contribution of this communication is the method developed to evaluate the matrix of cross-correlation coefficients between the 2^p words of a binary set $b_1 \cdots b_p$. We consider the cases in which 1) the set of harmonic factors $\{\alpha_j\}$, $j = 1, \cdots, p$, is the same for all binary words, and 2) the set $\{\alpha_j\}$ varies for any given word.

The cross-correlation coefficients to be evaluated are given by

$$\lambda_{ij} = \sigma_{ji} \equiv (\sigma_i \sigma_j)^{-1} \int_{-\pi/2}^{+\pi/2} F_i(x) F_j(x)\, dx, \qquad (2)$$

where σ_i^2 and σ_j^2 are the energies per TPF given by

$$\sigma_i^2 = \int_{-\pi/2}^{+\pi/2} F_i^2(x)\, dx, \qquad \sigma_j^2 = \int_{-\pi/2}^{+\pi/2} F_j^2(x)\, dx; \qquad (3)$$

F_i is the ith waveform in an encoding $(\alpha_1 \cdots \alpha_p)$ and F_j is the jth waveform in the same encoding or another one $(\alpha_1' \cdots \alpha_p')$.

The λ_{ij} are well known to be measures of the detectability of signals when correlating detectors are used. The distance d_{ij}^2 between F_i and F_j is determined by the relation

$$d_{ij}^2 = \sigma_i^2 + \sigma_j^2 - 2\sigma_i\sigma_j\lambda_{ij}. \tag{4}$$

In those cases where $\sigma_i^2 = \sigma_j^2$ (i.e., in the equi-energy waveforms sets such as we consider here), the distance equation reduces to

$$d_{ij}^2 = 2\sigma_i^2(1 - \lambda_{ij}). \tag{5}$$

In nonrigorous terms, a "good" set of waveforms is one in which 1) the half-distance between the closest members is enough greater than the length of a typical noise vector to keep the error rate manageably low, while 2) the number of signals available in the set is large enough to keep the data rate justifiably high under the given constraints of channel capacity and cost.

Approaches to calculation of cross-correlation coefficients

The most direct approach to evaluating the integrals defining λ_{ij} is to substitute trigonometric sums for the products. These sums are of only four possible types. They contain

1) only cosine terms with even-numbered harmonics,
2) only sine terms with even-numbered harmonics,
3) only cosine terms with odd-numbered harmonics, or
4) only sine terms with odd-numbered harmonics.

For, as can be seen, a TPF, $F(x)$, must equal either $\pm F(-x)$, and at the same time, $F(x)$ must equal either $\pm F(x + \pi)$. Therefore, $F(x)$ must expand to

$$F(x) = (2^{p-1})^{-1} \sum_{i=1}^{2^{p-1}} (\pm) \cos a_i x$$

or

$$F(x) = (2^{p-1})^{-1} \sum_{i=1}^{2^{p-1}} (\pm) \sin a_i x, \tag{6}$$

where the a_i are either all even or all odd, and $a_{2^{p-1}} = \alpha_p \pm \alpha_{p-1} \pm \cdots \pm \alpha_1$.

An ordering for the a_i is then defined by

$$a_i = a_1 + 2(n_1\alpha_1 + n_2\alpha_2 + \cdots + n_{p-1}\alpha_{p-1}), \tag{7}$$

where $a_1 = \min a_i = \alpha_p - (\alpha_{p-1} + \cdots + \alpha_1)$, and $n_i = 0, 1$. The decimal value of the ordering index i is given by the equivalent binary number $n_{p-1} \cdots n_1$. It should be observed that the 2^{p-1} values of $|a_i|$ are not necessarily distinct and nonzero—a fact that complicates the calculation of the λ_{ij} and also bears on the signal detectability and bandwidth.

- *Calculation of λ_{ij} by use of Parseval's theorem*

It can be shown with the help of Parseval's theorem that when $\{a_k\}$ for two product waveforms F_i and F_j are all odd or all even, the correlation integral can be taken between the limits $(-\pi, +\pi)$ as follows:

$$\int_{-\pi/2}^{+\pi/2} F_i(x)F_j(x)\,dx = \tfrac{1}{2}\int_{-\pi}^{+\pi} F_i(x)F_j(x)\,dx. \tag{8}$$

This certainly is the case when F_i and F_j have the same set of harmonic factors $\{\alpha_k\}$; for then, $\{a_k\}$ is the same for both. As a result, in such cases Parseval's theorem applies and the integral is calculable as the accumulated products of amplitudes of like harmonics at common frequencies $|a_k|$ in both expansions.

If, further, all the $|a_k|$ are distinct and nonzero, then corresponding amplitudes in F_i and F_j can differ only in polarity. The value of λ_{ij} then is readily calculated from

$$\lambda_{ij} = m_{ij}/2^{p-1}, \tag{9}$$

where m_{ij} equals the number of agreements in polarity on like harmonics minus the number of disagreements.

Equation (9) is similar to the correlation between binary code words in a $(2^{p-1}, p - 1)$ code. [The conventional notation "(n, k) code" means that each word has n bits of which k are data bits and $n - k$ are error-check bits.] The similarity arises since the product encoding in effect uses one bit to determine whether the sum expansion will be all sines or all cosines, and uses $p - 1$ bits to determine the polarity of the harmonics.

For waveforms F_i and F_j belonging to the same block, m_{ij} equals zero for all $i \neq j$. In the "classical" block, where $\{\alpha_i\} = 1, 2, 4, 8, \cdots$, the product encoding transforms binary words $b_1 \cdots b_r$ into orthogonal codes transmitted on 2^{p-1} subcarriers spaced uniformly apart, all subcarriers being sine or cosine on any one word. Figure 2(a) shows the waveforms of all 16 TPF's of the classical orthogonal set of order four. Figure 2(b) shows the energy density spectra of these same 16 TPF's. Table 1 gives the matrix of cross-correlation coefficients for the non-orthogonal block, $\{\alpha_i\} = 1, 2, 3, 4$.

When the Parseval expansion for λ_{ij} fails, or when the $|a_i|$ are non-distinct, assignment of TPF's to binary data will result in a non-binary, non-equi-energy encoding.

- *Calculation of λ_{ij} when Parsevals' theorem does not apply*

When $F_i(x) = F_i\,(x + \pi)$ but $F_j\,(x) = -F_j\,(x + \pi)$, the integral for λ_{ij} cannot in general be taken from $-\pi$ to $+\pi$. A direct evaluation is still possible if one substitutes the sums for F_i and F_j. Only two basic cases arise; F_i and F_j are both even or both odd functions, and one has even numbered harmonics while the other has odd-numbered harmonics. (If F_i is even and F_j is odd then $\lambda_{ij} = 0$.) Thus the integral is calculable as a double sum (of 2^{p-1} terms in each index) of integrals

$$C_{rs} = \int_{-\pi/2}^{+\pi/2} \sin 2rx \sin (2s - 1)x\,dx \text{ or} \tag{10}$$

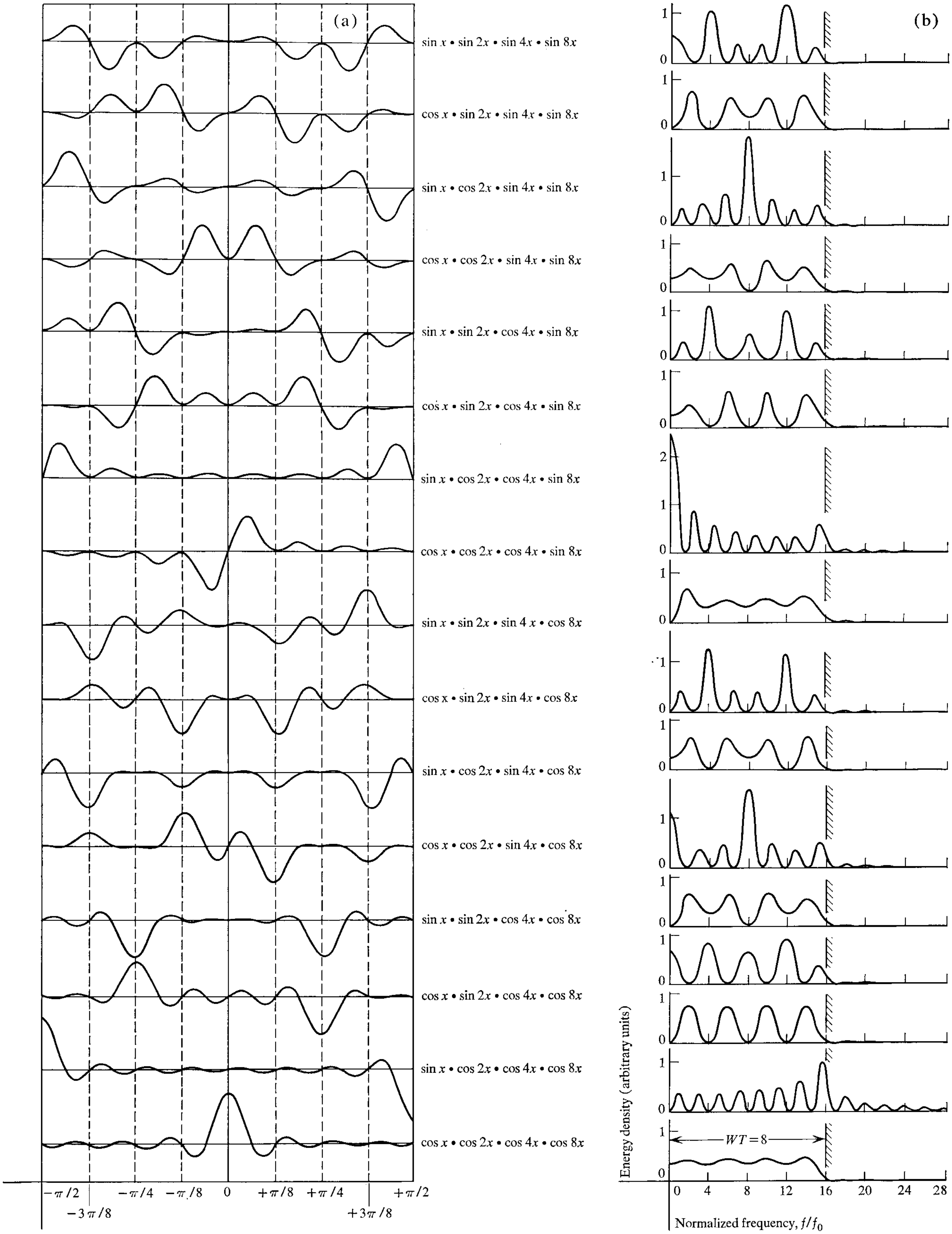

Figure 2 (a) Waveforms and (b) energy density spectra of the 16 TPF's of the "classical" orthogonal set of order 4.

Table 1 Cross-correlation matrix of TPF block where $\{a_i\} = 1, 2, 3, 4$. The notation C and S stands for a sine function and cosine function, respectively.

	q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
SSSS	1	1	0	0	$+\frac{1}{\sqrt{5}}$	0	$+\frac{1}{5}$	$-\frac{1}{\sqrt{5}}$	0	0	$-\frac{1}{\sqrt{5}}$	$+\frac{1}{\sqrt{65}}$	0	$-\frac{1}{3\sqrt{5}}$	0	0	$+\frac{1}{\sqrt{65}}$
CSSS	2	0	1	$+\frac{3}{\sqrt{77}}$	0	$+\frac{1}{11}$	0	0	$+\frac{3}{\sqrt{77}}$	$-\frac{3}{\sqrt{77}}$	0	0	$-\frac{1}{\sqrt{33}}$	0	$+\frac{1}{\sqrt{77}}$	$+\frac{1}{\sqrt{33}}$	0
SCSS	3	0	$+\frac{3}{\sqrt{77}}$	1	0	$-\frac{3}{\sqrt{77}}$	0	0	$+\frac{3}{7}$	$+\frac{1}{7}$	0	0	$-\frac{1}{\sqrt{21}}$	0	$+\frac{1}{7}$	$-\frac{3}{\sqrt{21}}$	0
CCSS	4	$+\frac{1}{\sqrt{5}}$	0	0	1	0	$+\frac{1}{\sqrt{5}}$	$+\frac{1}{3}$	0	0	$-\frac{1}{9}$	$-\frac{1}{3\sqrt{13}}$	0	$+\frac{1}{9}$	0	0	$+\frac{1}{\sqrt{13}}$
SSCS	5	0	$+\frac{1}{11}$	$-\frac{3}{\sqrt{77}}$	0	1	0	0	$-\frac{3}{\sqrt{77}}$	$-\frac{1}{\sqrt{77}}$	0	0	$+\frac{1}{\sqrt{33}}$	0	$+\frac{3}{\sqrt{77}}$	$-\frac{1}{\sqrt{33}}$	0
CSCS	6	$+\frac{1}{5}$	0	0	$+\frac{1}{\sqrt{5}}$	0	1	$-\frac{1}{\sqrt{5}}$	0	0	$+\frac{1}{3\sqrt{5}}$	$+\frac{1}{\sqrt{65}}$	0	$+\frac{1}{\sqrt{5}}$	0	0	$+\frac{1}{\sqrt{65}}$
SCCS	7	$-\frac{1}{\sqrt{5}}$	0	0	$+\frac{1}{3}$	0	$-\frac{1}{\sqrt{5}}$	1	0	0	$+\frac{1}{9}$	$-\frac{1}{\sqrt{13}}$	0	$-\frac{1}{9}$	0	0	$+\frac{1}{3\sqrt{13}}$
CCCS	8	0	$+\frac{3}{\sqrt{77}}$	$+\frac{3}{7}$	0	$-\frac{3}{\sqrt{77}}$	0	0	1	$+\frac{1}{7}$	0	0	$+\frac{3}{\sqrt{21}}$	0	$+\frac{1}{7}$	$+\frac{1}{\sqrt{21}}$	0
SSSC	9	0	$-\frac{3}{\sqrt{77}}$	$+\frac{1}{7}$	0	$-\frac{1}{\sqrt{77}}$	0	0	$+\frac{1}{7}$	1	0	0	$+\frac{1}{\sqrt{21}}$	0	$-\frac{5}{7}$	$-\frac{1}{\sqrt{21}}$	0
CSSC	10	$-\frac{1}{\sqrt{5}}$	0	0	$-\frac{1}{9}$	0	$+\frac{1}{3\sqrt{5}}$	$+\frac{1}{9}$	0	0	1	$+\frac{1}{3\sqrt{13}}$	0	$-\frac{5}{9}$	0	0	$+\frac{1}{3\sqrt{13}}$
SCSC	11	$+\frac{1}{\sqrt{65}}$	0	0	$-\frac{1}{3\sqrt{13}}$	0	$+\frac{1}{\sqrt{65}}$	$-\frac{1}{\sqrt{13}}$	0	0	$+\frac{1}{3\sqrt{13}}$	1	0	$-\frac{1}{3\sqrt{13}}$	0	0	$+\frac{1}{13}$
CCSC	12	0	$-\frac{1}{\sqrt{33}}$	$-\frac{1}{\sqrt{21}}$	0	$+\frac{1}{\sqrt{33}}$	0	0	$+\frac{3}{\sqrt{21}}$	$+\frac{1}{\sqrt{21}}$	0	0	1	0	$+\frac{1}{\sqrt{21}}$	$+\frac{1}{3}$	0
SSCC	13	$-\frac{1}{3\sqrt{5}}$	0	0	$+\frac{1}{9}$	0	$+\frac{1}{\sqrt{5}}$	$-\frac{1}{9}$	0	0	$-\frac{5}{9}$	$-\frac{1}{3\sqrt{13}}$	0	1	0	0	$-\frac{1}{3\sqrt{13}}$
CSCC	14	0	$+\frac{1}{\sqrt{77}}$	$+\frac{1}{7}$	0	$+\frac{3}{\sqrt{77}}$	0	0	$+\frac{1}{7}$	$-\frac{5}{7}$	0	0	$+\frac{1}{\sqrt{21}}$	0	1	$-\frac{1}{\sqrt{21}}$	0
SCCC	15	0	$\frac{1}{\sqrt{33}}$	$-\frac{3}{\sqrt{21}}$	0	$-\frac{1}{\sqrt{33}}$	0	0	$+\frac{1}{\sqrt{21}}$	$-\frac{1}{\sqrt{21}}$	0	0	$+\frac{1}{3}$	0	$-\frac{1}{\sqrt{21}}$	1	0
CCCC	16	$+\frac{1}{\sqrt{65}}$	0	0	$+\frac{1}{\sqrt{13}}$	0	$+\frac{1}{\sqrt{65}}$	$+\frac{1}{3\sqrt{13}}$	0	0	$+\frac{1}{3\sqrt{13}}$	$+\frac{1}{13}$	0	$-\frac{1}{3\sqrt{13}}$	0	0	1

$$C_{rs} = \int_{-\pi/2}^{+\pi/2} \cos 2rx \cos (2s - 1)x \, dx, \tag{11}$$

where r and s are integers such that $2r$ are harmonic numbers of F_i and $(2s - 1)$ are harmonic numbers of F_j. The result of evaluating the integral is $\sum \sum C_{rs}/(2^{p-1})^2$. Each C_{rs} is elementary; in fact,

$$C_{rs} = \frac{2A}{2r + 2s - 1} \mp \frac{2B}{2r - (2s - 1)} \tag{12}$$

in which

$A = 1$, if $2r + 2s - 1 = 1, 5, 9 \cdots$
$B = 1$, if $2r - (2s - 1) = 1, 5, 9 \cdots$
$A = -1$, if $2r + 2s - 1 = 3, 7, 11 \cdots$
$B = -1$, if $2r - (2s - 1) = 3, 7, 11 \cdots$

However, the number of terms is large and no general patterns are easily discernible. The formula can nevertheless be machine calculated.

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Received March 4, 1969
Revised June 1, 1970