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Minimum Polarized Distance Codes

Abstract: The choice of a code for a given application is influenced by many factors, such as economics, compatibility, and reliability. This paper is concerned solely with the reliability of codes, and shows how, for a given number of bits per character and a given minimum distance, the probability of undetected error in an asymmetric channel may be reduced by many orders of magnitude merely by the proper selection of coded characters. For a given minimum distance, an optimum selection of characters requires, as nearly as possible, the same number of "one" and "zero" bit failures to change one character to another. The concept of *polarized distance* is introduced, and it is shown how the probability of undetected error is related to the *minimum distance* of a code only in a symmetric channel, while the probability of undetected error is related to the *minimum polarized distance* in both symmetric and asymmetric channels.

The purpose of this paper is to present new theoretical concepts useful in the evaluation of codes, and not to recommend one code over another. The codes in the paper are used only as examples to illustrate the theoretical concepts involved.

Minimum distance

The *distance* between two coded characters is the number of bits that must change in one character so that the other character results. For example, the distance between the characters 11100 and 01010 is *three,* since the first, third, and fourth bits must change in order to go from one character to the other.

The *minimum distance* of a code is the minimum number of bits that must change in a coded character so that another valid character of the code will result.

In the hypothetical three-character code

A	1 1 1 0 0
B	0 1 0 1 0
C	1 0 1 0 1

the distance between A and B is *three;* between B and C, *five;* and between A and C, *two*. The minimum distance of this code is therefore *two*.

The relationship between the minimum distance and the amount of error detection and correction possible is

$M-1=C+D$ where $C\leqslant D$,

M is the minimum distance of a code, C is the number of bits in error that can be corrected, and D is the number of bits in error than can be detected.

Since no error can be corrected without being detected, C cannot be greater than D.

All possible values for C and D for values of M up to six are tabulated below:

M	*C*	*D*
1	0	0
2	0	1
3	0	2
	1	1
4	0	3
	1	2
5	0	4
	1	3
	2	2
6	0	5
	1	4
	2	3

Note that $D_{\max}=M-1$, while $C_{\max}\leqslant \dfrac{D_{\max}}{2}$,

both maximums, of course, being integers.

If a code is used for detection only, the minimum number of bits that must fail (change) so that an undetected error occurs is equal to the minimum distance of the code.

Codes with a minimum distance of *one* have no checking; the failure of one or more bits in a character can result in another valid character and therefore the error cannot be detected by a character check.

Codes with a minimum distance of *two* are single-error detecting codes. The failure of one bit in a character will always result in an invalid character, and therefore the error can be detected. The failure of two or more bits can result in another valid character, and therefore these errors cannot be detected.

Codes with a minimum distance of *three* are usually referred to as single-error correcting codes. The location of one bit in error can be determined and therefore the error can be corrected; the failure of two or more bits can appear to the error-correction system as a single error and an erroneous correction (undetected error) can result. If no correction is desired, double-error detection can be obtained with these codes. The failure of one or two bits in a character will always result in an invalid character and therefore the error can be detected. The failure of three or more bits, with or without correction, can result in another valid character and therefore these errors cannot be detected.

Codes with a minimum distance of *four* are usually referred to as single-error correcting double-error detecting codes. The location of one bit in error can be determined and therefore the error can be corrected; the failure of two bits can always be detected but their location cannot be determined for correction; the failure of three or more bits can appear to the error-correction system as a single error and an erroneous correction (undetected error) can result. If no correction is desired, triple-error detection can be obtained with these codes. The failure of one, two or three bits in a character will always result in an invalid character and therefore the error can be detected. The failure of four or more bits, with or without correction, can result in another valid character and therefore these errors cannot be detected.

The extension to codes of greater minimum distance should be obvious.

Probability of undetected error

To compare all codes on the same basis, it will be assumed that they are used for detection only and that a perfect check is applied. (A perfect check is one in which each character is checked to ascertain whether it is one of the valid characters in the code.)

The following symbols will be used:

P_1 probability of a *one* bit in error (dropping a bit),

P_0 probability of a *zero* bit in error (picking up a bit),

P_{max} the larger of P_1 and P_0,

P_{min} the smaller of P_1 and P_0,

P used to designate either P_1 or P_0 when $P_1 = P_0$,

P_u probability of undetected error,

$\stackrel{*}{=}$ "is of the order of magnitude of".

The following assumptions are made: the probability of bit error is small; bit errors occur randomly and are independent; all characters in a code occur randomly and are equally probable; the checking system will operate without failure.

Symmetric channels

In this section, only symmetric channels ($P_1 = P_0$) are considered.

For small values of P_1 and P_0, the following approximation can be shown to be valid:

$$P_u \approx \sum_{\substack{r=0\\s=M}}^{\substack{r=M\\s=0}} C_r P_1{}^r P_0{}^s$$

where r is the number of *one* bits in error and s is the number of *zero* bits in error, $r+s$ equaling the minimum distance, M, of the code; C_r is the average number of combinations per character in which these bits can cause an undetected error. This approximation implies that only the lowest-order undetected error need be considered in the evaluation of P_u.

More explicitly,

$$P_u \approx C_0 P_1{}^0 P_0{}^M + C_1 P_1{}^1 P_0{}^{M-1} + \ldots + C_M P_1{}^M P_0{}^0$$

Of major importance is the *order of magnitude* of P_u (as defined by $P_1{}^r P_0{}^s$); of only secondary importance is the value of the coefficient C_r. Therefore, in the discussion that follows, the C_r's will be ignored and only the orders of magnitude will be considered. (It is assumed that all C_r's are of the same order of magnitude.)

Thus

$$P_u \stackrel{*}{=} P_1{}^0 P_0{}^M + P_1{}^1 P_0{}^{M-1} + \ldots + P_1{}^M P_0{}^0 \,.$$

With some codes, some of the terms in the above expression equal zero; with other codes, all of the terms have some positive value. For instance, consider two single-error detecting ($M=2$) five-bit codes illustrated in Table 1. With Code A, where an undetected error can be caused by the failure of two *zero* bits, or one *one* bit and one *zero* bit, or two *one* bits,

$$P_u \stackrel{*}{=} P_1{}^0 P_0{}^2 + P_1{}^1 P_0{}^1 + P_1{}^2 P_0{}^0 \,,$$

while with Code B, where an undected error can be caused only by the failure of one *one* bit and one *zero* bit,

$$P_u \stackrel{*}{=} P_1{}^1 P_0{}^1 \,.$$

In both codes, in a symmetric channel, since $P_1 = P_0$, $P_u \stackrel{*}{=} P^2 = P^M$.

Asymmetric channels

These same two codes will now be considered in an asymmetric channel where $P_{max} \gg P_{min}$. The approximations in

Table 1 **Comparison of two single-error detecting five-bit codes with perfect check.**

Code A	*Decimal*	*Code B*
0 0 0 1 1	1	1 1 0 0 0
0 0 1 0 1	2	1 0 1 0 0
0 0 1 1 0	3	0 1 1 0 0
0 1 0 0 1	4	1 0 0 1 0
0 1 0 1 0	5	0 1 0 1 0
0 1 1 0 0	6	0 0 1 1 0
0 1 1 1 1	7	1 0 0 0 1
1 0 0 0 1	8	0 1 0 0 1
1 0 0 1 0	9	0 0 1 0 1
1 0 1 0 0	0	0 0 0 1 1
$M=2$		$M=2$
$K, L=0, 2$		$K, L=1, 1$
$P_u \doteqdot P_{\min}{}^K P_{\max}{}^L$ $=P_{\max}{}^2$		$P_u \doteqdot P_{\min}{}^K P_{\max}{}^L$ $=P_{\min} P_{\max}$

In a symmetric channel, $P^K_{\min} P^L_{\max}$ reduces to P^M, and $P^2_{\max}=P_{\min}P_{\max}=P^2$.

the previous section hold for these codes in an asymmetric channel. They do not hold for all codes in an asymmetric channel, however (see Appendix I).

With Code A, if $P_1 \ll P_0$, the probability of two *zero* bits failing is very much greater than the probability of one *one* bit and one *zero* bit failing or the probability of two *one* bits failing; thus $P_1{}^0 P_0{}^2$ is very much larger than $P_1{}^1 P_0{}^1$ or $P_1{}^2 P_0{}^0$, and

$$P_u \doteqdot P_1{}^0 P_0{}^2 = P_0{}^2 = P_{\max}{}^2 \, .$$

If $P_1 \gg P_0$, $P_1{}^2 P_0{}^2$ is very much larger than $P_1{}^1 P_0{}^1$ or $P_1{}^0 P_0{}^2$, and

$$P_u \doteqdot P_1{}^2 P_0{}^0 = P_1{}^2 = P_{\max}{}^2 \, .$$

Thus, if $P_{\max} \gg P_{\min}$,

$$P_u \doteqdot P_{\max}{}^2 \, .$$

On the other hand, with Code B, regardless of whether $P_1=P_0$, $P_1 \ll P_0$, or $P_1 \gg P_0$, an undetected error can be caused only if one *one* bit and one *zero* bit fail, and

$$P_u \doteqdot P_1{}^1 P_0{}^1 = P_{\min} P_{\max}$$

Comparing these two codes – both have the same number of bits per character, the same minimum distance, and the same check applied – it is found that the probability of undetected error with Code A is $P^2{}_{\max}/P_{\min}P_{\max} = P_{\max}/P_{\min}$ times the probability of undetected error with Code B.

As an example, assume $P_1=10^{-4}$ and $P_0=10^{-12}$. P_u for Code A is of the order of magnitude of $P_1{}^2=10^{-8}$, while P_u for Code B is of the order of magnitude of $P_1P_0=10^{-16}$; P_u for Code A is thus approximately 100,000,000 times that of Code B.

Note that the basic difference in the two codes is in the choice of the ten coded characters.

The key difference is that in Code A, one character may be changed to another character by the failure of only one type of bit (only *one*'s failing or only *zero*'s failing), while in Code B, coincident failures of both types of bits (both *one*'s and *zero*'s) are always required to change one character to another. For a given number of bits per character, a choice of coded characters always requiring failures of both types of bits, to change one character to another, will result in a lower P_u. (It should be noted that m-out-of-n codes satisfy this condition.) In fact, for a given minimum distance, a choice of characters requiring, as nearly as possible, the same number of each type of bit failure, maximizing the exponent of $P_{\min}$ and minimizing the exponent of $P_{\max}$, results in the minimum P_u.

Note also that the minimum distance is not an adequate description in an asymmetric channel: although both codes had the same value of M, Code A had a $P_u \doteqdot P_{\max}{}^2$ while Code B had a $P_u \doteqdot P_{\min}P_{\max}$. A new concept of code distance that will describe codes in both symmetric and asymmetric channels will therefore be introduced; this will be called the *polarized distance.*

Minimum polarized distance

The *polarized distance, k, l,* between two coded characters is the number of bits *of each type* that must change in one character so that the other character results. The polarized distance is thus made up of two numbers whose sum is equal to the distance between the characters. The convention will be established that if the two numbers are not equal, the smaller will be written first, that is, $k \leqslant l$.

For example, the polarized distance, k, l between the characters 11100 and 01010 is *1, 2*; in going from the first character to the second, one *zero* bit and two *one* bits must change; in going from the second character to the first, one *one* bit and two *zero* bits must change.

The *minimum polarized distance(s)* is obtained as follows:

Consider the distances between all possible pairs of characters in the code. Starting with the minimum distance, arrange these distances in increasing order.

For each distance, consider only the corresponding polarized distance with the *minimum k*. Leave these polarized distances in the same relative order as their corresponding distances.

The first polarized distance is a (the) *minimum polarized distance*. This will be symbolized by K, L. $(K+L=M)$.

The next polarized distance *with a* $k<K$, if any, is also a minimum polarized distance. This will be symbolized by K', L'. $(K'+L'>K+L)$.

Table 2 **Polarized distance chart Code A ($K, L=0, 2$).**

From ↓ / *To* →	0 10100	1 00011	2 00101	3 00110	4 01001	5 01010	6 01100	7 01111	8 10001	9 10010
0 10100		2, 2	1, 1	1, 1	2, 2	2, 2	1, 1	1, 3	1, 1	1, 1
1 00011	2, 2		1, 1	1, 1	1, 1	1, 1	2, 2	**0, 2**	1, 1	1, 1
2 00101	1, 1	1, 1		1, 1	1, 1	2, 2	1, 1	**0, 2**	1, 1	2, 2
3 00110	1, 1	1, 1	1, 1		2, 2	1, 1	1, 1	**0, 2**	2, 2	1, 1
4 01001	2, 2	1, 1	1, 1	2, 2		1, 1	1, 1	**0, 2**	1, 1	2, 2
5 01010	2, 2	1, 1	2, 2	1, 1	1, 1		1, 1	**0, 2**	2, 2	1, 1
6 01100	1, 1	2, 2	1, 1	1, 1	1, 1	1, 1		**0, 2**	2, 2	2, 2
7 01111	1, 3	**0, 2**	**0, 2**	**0, 2**	**0, 2**	**0, 2**	**0, 2**		1, 3	1, 3
8 10001	1, 1	1, 1	1, 1	2, 2	1, 1	2, 2	2, 2	1, 3		1, 1
9 10010	1, 1	1, 1	2, 2	1, 1	2, 2	1, 1	2, 2	1, 3	1, 1	

The next polarized distance *with a* $k<K'$, if any, is also a minimum polarized distance. This will be symbolized by K'', L''. $(K''+L''>K'+L')$.

This process is continued until the minimum polarized distance with the absolute minimum k is obtained.

In most codes, there will be only one minimum polarized distance, K, L, (K being the absolute minimum k). In the discussion that follows, only this case will be considered. (The case of codes with multiple minimum polarized distances will be discussed in Appendix I.)

For an example, refer to Code A polarized distance chart in Table 2. There are only two distances between the possible pairs of characters: *2* and *4*. The corresponding polarized distances with the minimum k, in each case, are 0, 2 and 1, 3 respectively. The minimum polarized distance, K, L of this code is thus 0, 2.

With the 2-out-of-5 code, the polarized distances with the minimum k are 1, 1 and 2, 2. Thus, the minimum polarized distance, K, L, of Code B is 1, 1.

In a symmetric channel, the minimum distance of a code is related to the probability of undetected error as follows:

$$P_u \doteqdot P^M .$$

In an asymmetric channel the minimum polarized distance of a code is related to the probability of undetected error as follows:

$$P_u \doteqdot P_{\min}{}^K P_{\max}{}^L .$$

This expression also applies in a symmetric channel, where $P_{\min}=P_{\max}=P$, the expression reducing to $P_u \doteqdot P^K P^L = P^{K+L}=P^M$.

(The order of magnitude of P_u can be expressed in the following form:

$$P_u \doteqdot P_{\min}{}^0 P_{\max}{}^M + P_{\min}{}^1 P_{\max}{}^{M-1} + P_{\min}{}^2 P_{\max}{}^{M-2} + \ldots$$

In an asymmetric channel where $P_{\max} \gg P_{\min}$, the first term is the largest, with each succeeding term diminishing in order of magnitude. Therefore, P_u is of the order of magnitude of the first non-zero term encountered. The method of obtaining K, L in effect specifies this term.)

Optimum minimum polarized distance

From the expression $P_u \doteqdot P_{\min}{}^K P_{\max}{}^L$ it can be seen that, *for a given M*, P_u is a minimum when the coded characters are chosen so that K is a maximum and L is a minimum. Therefore, *for a given M*, $K_{\max} L_{\min}$ defines the optimum minimum polarized distance. For instance, comparing some codes with $M=5$:

If $K, L=0, 5$, $\quad P_u \doteqdot P_{\max}{}^5$

If $K, L=1, 4$, $\quad P_u \doteqdot P_{\min} P_{\max}{}^4$

If $K, L=2, 3$, $\quad P_u \doteqdot P_{\min}{}^2 P_{\max}{}^3$

The last case illustrates the optimum minimum polarized distance ($K, L=2, 3$) for $M=5$.

It is obvious that if $P_{max} \gg P_{min}$, the P_u of the second code can be orders of magnitude lower than that of the first code, and the P_u of the third code can be orders of magnitude lower than that of the first and second code.

Thus, two codes may have the same number of bits per character, the same minimum distance, and the same check applied, and yet the probability of undetected error in an asymmetric channel may be orders of magnitude lower in one of the codes, owing to the choice of coded characters. If the characters are chosen so as to maximize K and minimize L (requiring, as nearly as possible, the same number of each type of bit failure to change one character to another), a code with a minimum P_u will result.

It should be noted that if a character consisting of all *one*'s or all *zero*'s is chosen, $K=0$, and the *least* desirable minimum polarized distance results.

In a highly asymmetric channel, the probability of undetected error of one code may be orders of magnitude lower than that of another code having a greater minimum distance. (This will be discussed in Appendix II.)

Maximum number of coded characters

If, for a specified number of bits per character and a specified minimum distance, the coded characters are chosen so as to obtain the optimum minimum polarized distance, it should be expected that a smaller number of possible characters may result than would otherwise. The following questions are therefore logically raised: given a specified number of bits per character and a specified minimum distance, what is the maximum number of coded characters that can be obtained? What is the maximum number of coded characters that can be obtained for the various possible minimum polarized distances, and, in particular, for the optimum minimum polarized distance?

The first question has been examined by Hamming,[1] Plotkin,[2] and Joshi.[3] There is no general formula for evaluating these maximums; the references contain many formulas giving upper bounds for maximums and actual values for certain cases.

Table 3 shows the known values of the maximum number of coded characters for up to eleven bits per character and up to a minimum distance of six.

Given two of the three variables – minimum distance, number of bits per character, and maximum number of coded characters – the third can be obtained from the table.

For example, the table shows that twelve characters are possible with a ten-bit double-error correcting double-error detecting (minimum distance five) code.

Or given a maximum of six bits per character available and a ten-character (numeric) code required, what is the best check that can be obtained? The table shows that a minimum distance three code allows only eight characters; therefore, at best only a minimum distance two (single-error detecting) code can be obtained. However, with a minimum distance two code, six bits per character would not be required to give the desired ten characters; five bits per character allow sixteen characters which is sufficient.

Table 3 **Maximum number of coded characters.**

No. of bits per character	*Minimum distance* 1	2	3	4	5	6
1	2					
2	4	2				
3	8	4	2			
4	16	8	2	2		
5	32	16	4	2	2	
6	64	32	8	4	2	2
7	128	64	16	8	2	2
8	256	128		16	4	2
9	512	256			6	4
10	1,024	512			12	6
11	2,048	1,024			24	12

If a ten-character single-error correcting (minimum distance three) code is desired, the table shows that seven bits per character are required.

For each minimum distance, the minimum entry satisfying the character requirements for a numeric code (ten characters) has been circled.

The second question is as yet unanswered, and the completion of a table such as Table 3, with each minimum distance replaced by all possible corresponding minimum polarized distances, is a suggestion for future study.

A portion of such a table is shown in Table 4, where values for optimum minimum polarized distances corresponding to the circled entries in Table 3 are given. (These entries are termed "possible" rather than "maximum" since no proof is given that they are maximum.)

These entries will now be discussed.

For $M=1$, the concept of minimum polarized distance is trivial, and K, L must equal 0, 1.

For $M=2$, the optimum K, L equals 1, 1. An example of a numeric code having this minimum polarized distance is Code B. For such a code, $P_u \doteq P_{min}P_{max}$. Code A has the same number of bits per character, but a $P_u \doteq P_{max}^2$. In an asymmetric channel, the probability of undetected error with Code A is P_{max}/P_{min} times that of Code B.

For $M=3$, the optimum K, L equals 1, 2. An example satisfying the numeric code requirements follows:

1 1 1 1 0 0 0
1 1 0 0 1 1 0
1 1 0 0 0 0 1
1 0 1 0 1 0 1
1 0 1 0 0 1 0
0 1 1 0 0 1 1
0 1 1 0 1 0 0
0 0 0 0 1 1 1
0 0 1 1 0 0 1
0 0 1 1 1 1 0
0 1 0 1 0 1 0
0 1 0 1 1 0 1
1 0 0 1 1 0 0
1 0 0 1 0 1 1

Any ten of the fourteen characters can be chosen for a numeric code. The $P_u \overset{*}{=} P_{\min} P_{\max}^2$. The seven-bit Hamming code has the same number of bits per character, but a $P_u \overset{*}{=} P_{\max}^3$. The P_u for the Hamming code is $P_{\max}/P_{\min}$ times that for the code illustrated.

For $M=4$, the optimum K, L equals 2, 2. An example satisfying the numeric code requirements follows:

1 1 1 1 0 0 0 0
1 1 0 0 1 1 0 0
1 1 0 0 0 0 1 1
1 0 1 0 1 0 1 0
1 0 1 0 0 1 0 1
0 1 1 0 0 1 1 0
0 1 1 0 1 0 0 1
0 0 0 0 1 1 1 1
0 0 1 1 0 0 1 1
0 0 1 1 1 1 0 0
0 1 0 1 0 1 0 1
0 1 0 1 1 0 1 0
1 0 0 1 1 0 0 1
1 0 0 1 0 1 1 0

Again any ten of the fourteen characters can be chosen for a numeric code. The $P_u \overset{*}{=} P_{\min}^2 P_{\max}^2$. The eight-bit Hamming code has the same number of bits per character, but a $P_u \overset{*}{=} P_{\max}^4$. The P_u for the Hamming code is $(P_{\max}/P_{\min})^2$ times that for the code shown.

(The seven-bit code previously illustrated was obtained from this code by the elimination of the eighth bit.)

Table 4 **Possible number of coded characters.**

No. of bits per character	*Optimum minimum polarized distance* 0, 1	1, 1	1, 2	2, 2	2, 3	3, 3
1						
2						
3						
4	16					
5		10				
6						
7			14			
8				14		
9						
10					11	
11						11

For $M=6$, the optimum K, L equals 3, 3. An example satisfying the numeric code requirements follows:

1 0 1 1 1 0 0 0 1 0 1
0 1 1 1 0 0 0 1 0 1 1
1 1 1 0 0 0 1 0 1 1 0
1 1 0 0 0 1 0 1 1 0 1
1 0 0 0 1 0 1 1 0 1 1
0 0 0 1 0 1 1 0 1 1 1
0 0 1 0 1 1 0 1 1 1 0
0 1 0 1 1 0 1 1 1 0 0
1 0 1 1 0 1 1 1 0 0 0
0 1 1 0 1 1 1 0 0 0 1
1 1 0 1 1 1 0 0 0 1 0

Any ten of the eleven characters can be chosen for a numeric code. The $P_u \overset{*}{=} P_{\min}^3 P_{\max}^3$. The P_u for a $K, L=0$, 6 code $\overset{*}{=} P_{\max}^6$, being $(P_{\max}/P_{\min})^3$ times that for the code shown. By the elimination of any one column of the code illustrated, a ten-bit code with the optimum $K, L=2$, 3 can be obtained.

Appendix I: Codes with multiple minimum polarized distances

Although most codes have only one minimum polarized distance, it is possible to construct codes having multiple minimum polarized distances.

For example, the following hypothetical four-character code is analyzed:

A	0 0 0 0 0 0 0 0
B	0 0 0 1 1 1 1 1
C	0 1 1 0 0 1 1 1
D	1 1 1 1 1 1 1 0

Between	*Distance*	*Polarized distance*
A and B	5	0, 5
A and C	5	0, 5
A and D	7	0, 7
B and C	4	2, 2
B and D	4	1, 3
C and D	4	1, 3

The distances are arranged in increasing order, along with each corresponding polarized distance with the minimum k:

Distance	*Corresponding polarized distance with minimum k*
4	1, 3
5	0, 5
7	0, 7

The first minimum polarized distance, K, L, equals 1, 3.

The next polarized distance, 0, 5, has a $k<K(0<1)$; therefore, it is also a minimum polarized distance, symbolized by K', L'.

The next polarized distance, *0, 7*, does not have a $k<K'$; therefore, it is not a minimum polarized distance.

Thus there are two minimum polarized distances for this code:

$$K, L=1, 3$$

and

$$K', L'=0, 5$$

If there are two minimum polarized distances, as in this example, K, L and the relationship $P_u \overset{*}{=} P_{\min}^K P_{\max}^L$ apply from symmetry to a certain ratio of asymmetry, while K', L' and the relationship $P_u \overset{*}{=} P_{\min}^{K'} P_{\max}^{L'}$ apply from this ratio of asymmetry to all higher ratios of asymmetry.

In codes of this type, the following relationships exist:

$$K+L=M$$

$$K'+L'>M$$

$$K'+L'>K+L$$

$$K>K'$$

$$L'>L$$

$$L'-L>K-K'$$

The relationship $P_u \overset{*}{=} P_{\min}^K P_{\max}^L$ applies in the lower ranges of asymmetry where $P_{\min}^K P_{\max}^L \geqslant P_{\min}^{K'} P_{\max}^{L'}$, or when $P_{\min}^{K-K'} \geqslant P_{\max}^{L'-L}$.

The relationship $P_u \overset{*}{=} P_{\min}^{K'} P_{\max}^{L'}$ applies in the higher ranges of asymmetry where $P_{\min}^{K-K'} \leqslant P_{\max}^{L'-L}$.

In the preceding example, $P_u \overset{*}{=} P_{\min}^K P_{\max}^L$ equal to $P_{\min} P_{\max}^3$ applies when $P_{\min}^{1-0} \geqslant P_{\max}^{5-3}$, or when $P_{\min} \geqslant P_{\max}^2$. The relationship $P_u \overset{*}{=} P_{\min}^{K'} P_{\max}^{L'}$ equal to $P_{\max}^5$ applies when $P_{\min} \leqslant P_{\max}^2$.

It is also possible to construct codes in which there are more than two minimum polarized distances. For example, a code can be constructed with the following minimum polarized distances:

$$K,\ L=2, 2$$

$$K',\ L'=1, 4$$

$$K'',\ L''=0, 7$$

The relationships between K, L and K', L' are similar to those between K', L' and K'', L''; these relationships, of course, extend to any number of minimum polarized distances.

Minimum polarized distance K, L and $P_u \overset{*}{=} P_{\min}^2 P_{\max}^2$ apply in the lowest ranges of asymmetry where $P_{\min}^{2-1} \geqslant P_{\max}^{4-2}$, or when $P_{\min} \geqslant P_{\max}^2$.

Minimum polarized distance K', L' and $P_u \overset{*}{=} P_{\min} P_{\max}^4$ apply when $P_{\min} \leqslant P_{\max}^2$ and when $P_{\min}^{1-0} \geqslant P_{\max}^{7-4}$; that is, when $P_{\max}^3 \leqslant P_{\min} \leqslant P_{\max}^2$.

Minimum polarized distance K'', L'' and $P_u \overset{*}{=} P_{\max}^7$ apply in the highest ranges of asymmetry where $P_{\min} \leqslant P_{\max}^3$.

It is possible for a minimum polarized distance not to apply in any range. For example, in a code with the following minimum polarized distances,

$$K,\ L=2, 2$$

$$K',\ L'=1, 4$$

$$K'',\ L''=0, 6$$

minimum polarized distance K, L and $P_u \overset{*}{=} P_{\min}^2 P_{\max}^2$ apply when $P_{\min} \geqslant P_{\max}^2$; K', L' and $P_u \overset{*}{=} P_{\min} P_{\max}^4$ apply when $P_{\max}^2 \leqslant P_{\min} \leqslant P_{\max}^2 (?)$; K'', L'' and $P_u \overset{*}{=} P_{\max}^6$ apply when $P_{\min} \leqslant P_{\max}^2$. Thus, K, L applies when $P_{\min} \geqslant P_{\max}^2$ and K'', L'' applies when $P_{\min} \leqslant P_{\max}^2$, K', L' not applying in any range. (K', L' can be considered as applying in the case of $P_{\min} = P_{\max}^2$, but this is trivial since in this case K, L and K'', L'' also apply, and $P_{\min}^2 P_{\max}^2 = P_{\min} P_{\max}^4 = P_{\max}^6$).

Appendix II: Comparison of codes with different minimum distances

Codes with different minimum distances can be compared using the principles discussed in Appendix I, and the following observations can be made: in a highly asymmetric channel, the probability of undetected error of one code may be orders of magnitude lower than that of another code having a greater minimum distance.

For example, compare the single-error detecting ($M=2$) five-bit Code B with the double-error detecting ($M=3$) seven-bit Hamming code. For Code B, $K, L=1, 1$, and $P_u \doteqdot P_{\min} P_{\max}$; for the Hamming code, $K, L=0, 3$, and $P_u \doteqdot P_{\max}^3$.

In the lower ranges of asymmetry, when $P_{\min}^{1-0} > P_{\max}^{3-1}$, or when $P_{\min} > P_{\max}^2$, the P_u for the Hamming code is lower ($P_{\min} P_{\max} > P_{\max}^3$). However, in the higher ranges of asymmetry, when $P_{\min} < P_{\max}^2$, the P_u for Code B is lower ($P_{\min} P_{\max} < P_{\max}^3$).

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