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Periodic Solutions of the Wave Equation with a Nonlinear Interface Condition

Abstract: In this paper we consider the problem of the voltage oscillations in a transmission line when a diode represented as a nonlinear capacitance is placed in shunt in that line. In particular we consider the response of this line to periodic driving voltages and study the periodic responses. This physical situation is shown to lead to the mathematical model of the wave equation in the voltage in a domain with an internal boundary (the interface), at which the voltage is required to satisfy a nonlinear jump condition. By an application of Gauss' theorem, the problem is reduced to a nonlinear difference-differential equation. In the case that the generator driving the line is matched to it, this family of equations reduces to a family of nonlinear differential equations. The paper is concerned with a study of the periodic solutions of these two classes of equations.

I. Introduction and derivation of model

• *1. Introduction and description of contents*

In this paper we will consider the propagation of voltage waves down a transmission line when a diode is placed in shunt in that line. We show that this state of affairs can be described by a boundary value problem for the wave equation for the voltage across the line coupled with a nonlinear voltage drop across the capacitor. Thus this problem bears certain similarities to the problem of the bowing of a violin string treated by Lord Rayleigh[1] and J. B. Keller.[2]

In Section 2 we make use of the fact that a diode can be characterized as a capacitor, in which the capacitance is a function of the voltage drop, to derive the wave equation model for our problem. We then show by an application of Gauss's theorem to solutions of the wave equation that the boundary value problem is equivalent to solving a nonlinear difference-differential equation (2.22). In the special case where the generator driving the line is matched to the line, this functional equation reduces to a nonlinear differential equation (2.23). In Section 3 to Section 6 we discuss the properties of solutions of the differential equation. Section 3 and Section 4 are coupled into Part II of this paper. This part uses the topological techniques of phase plane analysis and Banach space methods to deduce general qualitative properties of solutions of the differential equation.

In Section 3 we use phase plane methods to demonstrate the existence, uniqueness and stability of a periodic solution of the differential equation. We then point out the consequence that the voltage variation of a lumped R-C circuit can have neither subharmonic nor superharmonic response to a harmonic voltage source. A special case of voltage variation of the nonlinear capacitance which is of interest (capacitance $=c(1+\rho v)$, where v is voltage, and c and ρ are constants), is then discussed by making use of singular point analysis.

In Section 4 we consider the question of existence, uniqueness and stability of a periodic solution in the special case, $c(v)=c(1+\rho v)$, just mentioned. Because of the presence of singular points of the differential equation, we are obliged to resort to certain Banach space methods to deduce these results. The methods used are the implicit function theorem and the Picard Fixed Point Theorem. We point out that this discussion motivates and justifies the use of a perturbational procedure employed in Part III of this paper.

In Part III we continue our analysis of the special case, $c(v)=c(1+\rho v)$, by means of perturbational methods. In Section 5 we use these methods to obtain the response curve (amplitude of the harmonic output versus applied frequency) of the periodic solution in response to a harmonic driving voltage. We then sketch a plausibility argument for the nonexistence of subharmonic response and point out how such responses can be achieved by modifications of the circuit under discussion.

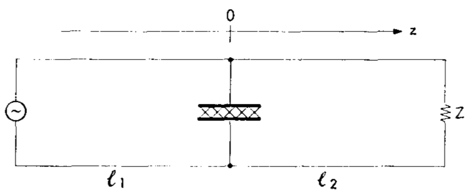
In Section 6 we consider the question of amplification and conversion. Thus we drive our circuit with a combination of harmonic inputs and discuss the transfer of energy from the components of the input to the components of the output. By the perturbational procedure we deduce the extent to which this device (commonly described as a carrier and signal combination) is of practical importance. We then derive the response relation pertaining to this situation.

In Part III we turn to the nonlinear difference-differential equation (2.22). In Section 7 we consider the case when the period T and the lag τ are commensurable. In this case we reduce the solving of the difference-differential equation to solving successively two simpler equations. The first of these is a linear difference equation and the second is a transcendental equation. In Section 8 we use perturbational methods to derive the response relations for periodic solutions in the general case when there is no commensurability restriction.

In Section 2 we state our problem and derive the mathematical model used to describe it.

• 2. *Statement of the problem and derivation of the mathematical model*

A crystal diode placed in a transmission line behaves in many respects as a capacitor in shunt in that transmission line, provided that the capacitance is taken to be a function of the voltage across it.[3] Consider then a segment of transmission line of length l_1+l_2. At the left end of the



line, let there be a generator, while at the right end, let the line be terminated in an impedance Z. At distance l_1 from the generator (at $x=0$, in the figure) let a capacitor, of capacitance $c(v)$, be placed in shunt. If $v(x, t)$ and $i(x, t)$ are, respectively, the voltage across the line and the current flowing down the line at position x and time t, we have the transmission line equations for a lossless line

$$\frac{\partial v}{\partial x} = -L\frac{\partial i}{\partial t} \tag{2.1}$$

$$\frac{\partial i}{\partial x} = -C\frac{\partial v}{\partial t}\,. \tag{2.2}$$

Here L and C are, respectively, the series inductance per unit length of line and the shunt capacitance per unit length of the line. Combining (2.1) and (2.2) gives

$$\frac{\partial^2 v}{\partial x^2} = LC\,\frac{\partial^2 v}{\partial t^2}\,. \tag{2.3}$$

At $x=0$, the current flowing down the line suffers a jump equal to the current flowing through the capacitor. If we use the notation $[f]=f(0+)-f(0-)$ to denote the jump in a quantity at $x=0$, we have[4]

$$[i]=q_t=\frac{dq}{dv}\,\frac{dv}{dt}=-c(v)v_t\,, \tag{2.4}$$

where $q(t)$ is the charge on the capacitor. Differentiating (2.4) with respect to t yields

$$[i_t]=-(c(v)v_t)_t\,. \tag{2.5}$$

Combining (2.1) and (2.5), gives

$$[v_x]=L(c(v)v_t)_t\,. \tag{2.6}$$

At $x=-l_1$, we have

$$v=\phi(t)\,, \tag{2.7}$$

where $\phi(t)$ denotes the input voltage at the generator. At $x=l_2$, we have the impedance boundary condition[5]

$$v_x+CZv_t=0\,. \tag{2.8}$$

If we denote by $f(x)$ and $-g(x)/C$ the voltage across the line and the current gradient down the line at time $t=0$, we have the following boundary value problem with a nonlinear interface condition for $v(x, t)$.

$$v_{xx}=\frac{1}{a_0^2}\,v_{tt},\ a_0=(LC)^{-1/2},\qquad t>0,\ -l_1<x<l_2,\ x\neq 0\,, \tag{2.9}$$

$$v(x,0)=f(x),\qquad v_t(x,0)=g(x)\,, \tag{2.10}$$

$$v_x+\alpha v_t=0,\qquad \alpha=CZ,\ x=l_2\,, \tag{2.11}$$

$$v=\phi(t),\qquad x=-l_1\,, \tag{2.12}$$

$$[v_x]=\frac{2}{a_0}\,(\bar{c}(v)v_t)_t,\ \bar{c}(v)=\frac{a_0L}{2}\,c(v),\ x=0\,. \tag{2.13}$$

We will now drop the bar in (2.13) and show that the boundary value problem (2.9)–(2.13) is equivalent to a nonlinear functional equation.

If D is a domain in $x-t$ space in which $v(x, t)$ is a solution of the wave equation, then Gauss's theorem yields

$$0=\iint_D (a_0^2v_{xx}-v_{tt})\,dxdt$$

$$=\oint[a_0^2v_x\cos(n,x)-v_t\cos(n,t)]\,ds\,. \tag{2.14}$$

The integral in the right member of (2.14) is a line integral taken around the boundary B of D. Then s, the arc length of B, is taken as increasing if one traverses B with D lying to the left. $\operatorname{Cos}(n, x)$ and $\cos(n, t)$ are the direction cosines of the exterior normal to B.

We apply the formula (2.14) to the two domains indicated in Fig. 1. The domain bounded by $P, A_1, P_1, A_2, \ldots A_n, B, 0, P$ and the domain bounded by $P, 0, l_2, D, P$. The diagonal lines in the figures are segments of characteristics of the wave equation, i.e., lines with slope $\pm a_0^{-1}$.

Since we are ultimately interested in the steady state we will assume that $f(x)=g(x)=0$. Now application of

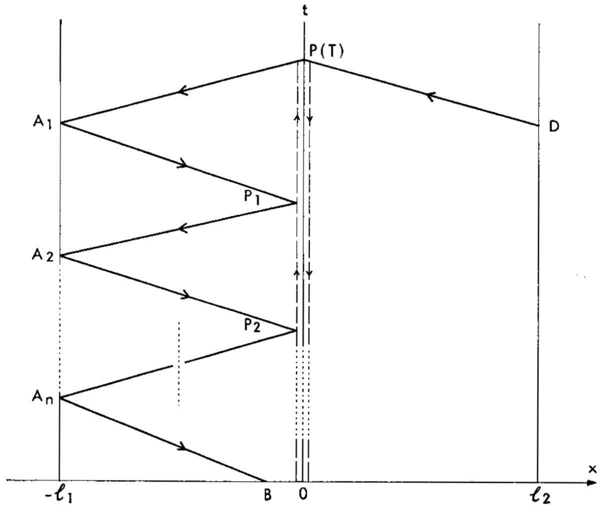


Figure 1

(2.14) to the two domains in question yields respectively:

$$\int_0^T a_0^2 v_x(0-,t)\,dt - a_0 v(P) + 2a_0 \sum_{j=1}^{n(T)} [\phi(A_j) - v(P_j)] = 0 \tag{2.15}$$

$$\int_T^0 a_0^2 v_x(0+,t)\,dt - a_0 v(P) + a_0 v(D) + \int_0^D a_0^2 v_x(l_2,t)\,dt = 0\,. \tag{2.16}$$

Adding (2.15) and (2.16) gives

$$-\int_0^T [v_x]\,dt = \frac{2}{a_0} v(P) + \frac{2}{a_0} \sum_{j=1}^{n(T)} [v(P_j) - \phi(A_j)] - \frac{v(D)}{a_0} - \int_0^D v_x(l_2,t)\,dt\,. \tag{2.17}$$

If we insert (2.11) and (2.13) into (2.17), it becomes

$$\int_0^T -(c(v)v_t)_t\,dt = v(P) + \sum_{j=1}^{n(T)} [v(P_j) - \phi(A_j)] - 1/2[v(D) - a_0 \int_0^D v_t(l_2,t)\,dt]\,. \tag{2.18}$$

Performing the integrations in (2.18) gives

$$-c(v)v_t(P) = v(P) + \sum_{j=1}^{n(T)} [v(P_j) - \phi(A_j)] - 1/2 v(D)(1-\alpha a_0)\,. \tag{2.19}$$

The presence of the load Z at a distance l_2 from the diode will cause reflections of voltage waves to pass through the diode. Thus the effect of a voltage wave generated at $x=-l_1$ passing through the diode is not altered by the presence of Z. Since we are fundamentally interested in the effect of the nonlinearity on a voltage wave and not upon the complications due to the passage of reflections, we make the assumption that $l_2=\infty$ or equivalently that $1-\alpha a_0 = 1 - Z(C/L)^{1/2} = 0$ (i.e., Z is matched to the line). Then (2.19) becomes

$$-c(v)v_t(P) = v(P) + \sum_{j=1}^{n(T)} [v(P_j) - \phi(A_j)]\,. \tag{2.20}$$

Now let

$$\tau = |P_j - P_{j-1}| = \frac{2l_1}{a_0} \tag{2.21}$$

$\tau/2$ is the time it takes an impulse produced at the generator to reach the capacitor.

If we write (2.20) with P replaced by P_1 (i.e., with T replaced by $T-\tau$), and subtract the result from (2.20), we obtain after replacing T by t, the following functional equation for $v(t)=v(0,t)$

$$c(v)v_t\Big|_{t-\tau}^{t} = -v(t) + \phi(t-1/2\tau)\,. \tag{2.22}$$

This equation may be further simplified by making an assumption based upon the above reasoning which led to setting $l_2=\infty$. If we assume that the generator is matched to the line or that $l_1=\infty$, (and therefore that $\tau=\infty$), and if we denote by $\bar{\phi}(t)$, the limit as $\tau\to\infty$ of $\phi(t-1/2\tau)$, (2.22) becomes, upon dropping the bar,

$$c(v)v_t = -v + \phi\,. \tag{2.23}$$

In this equation ϕ is the voltage generated at an infinite distance from the capacitor.

We remark that $v(x,t)$ in the infinite line is computable from $v(0,t)$ by the formula

$$2v(x,t) = v\left(0, t-\frac{x}{a_0}\right) + v\left(0, t+\frac{x}{a_0}\right). \tag{2.24}$$

Even in the case of the finite line $v(x,t)$ is readily computed once $v(0,t)$ is known. One method of finding $v(x,t)$ after $v(0,t)$ is known, is to take Laplace transforms. However, we will not dwell upon this matter, since the procedure is classical.

Our point of view in this paper will be to analyze the periodic solutions of the nonlinear equations (2.22) and (2.23). We consider what can be said in general about these equations when $\phi(t)$ is periodic and then specialize to the case when

$$c(v) = c(1+\rho v)\,, \tag{2.25}$$

with c a constant and ρ a parameter, and for various harmonic $\phi(t)$ such as

$$\phi(t) = F\cos\omega t\,. \tag{2.26}$$

and

$$\phi(t) = F_1\cos\omega_1 t + F_2\cos\omega_2 t\,. \tag{2.27}$$

In the following part of this paper we consider the questions of existence, uniqueness and stability of periodic solutions of (2.23).

II. Topological methods of studying the existence, uniqueness, and stability of periodic solutions of the differential equation

• *3. Phase plane analysis*

In this section we will study equation

$$\frac{dv}{dt}=\frac{-v+\phi}{c(v)}, \tag{3.1}$$

where $\phi(t)$ is a periodic function of time. We will investigate the questions of existence, uniqueness, and stability of periodic solutions of this equation.

Suppose first of all that $c(v)$ is a real physical capacitance so that $c(v)>0$ for all v. In this case we can enunciate the following:

• *Theorem*

Let $c(v)$ and $\phi(t)$ be continuous functions of their arguments. If $\phi(t)$ is periodic with period T and $c(v)>0$, then there exists one and only one periodic solution of the differential equation (3.1). Moreover, this solution has the period T and is stable in the sense that all solutions of (3.1) approach it as $t\to\infty$.

• *Proof*

(i) *Existence*

Since $c(v)>0$,

$$\text{signum}\left(\frac{dv}{dt}\right)=\text{signum }(\phi-v)\,. \tag{3.2}$$

Since $\phi(t)$ is a continuous periodic function, it has a maximum and a minimum which we denote respectively by V_M and V_m. Then by (3.2), (see Fig. 2),

$$\text{signum}\left(\frac{dv}{dt}\right)>0,\text{ for } v<V_m \tag{3.3}$$

and

$$\text{signum}\left(\frac{dv}{dt}\right)<0,\text{ for } v>V_M. \tag{3.4}$$

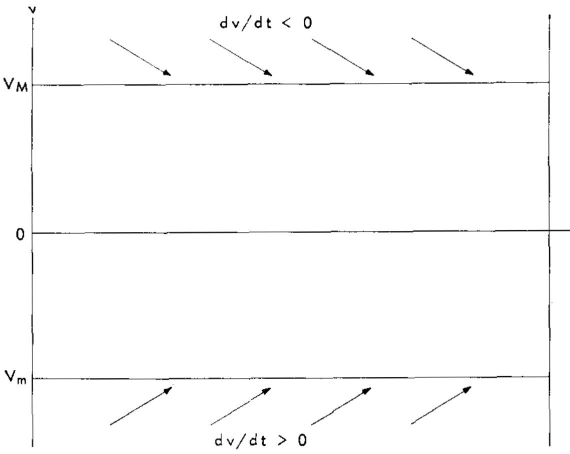


Figure 2

Thus any solution of (3.1) which at $t=0$ is less than or equal to V_M will reach $t=T$ at a value of $v\le V_M$, i.e.,

$$v(0)\le V_M\Rightarrow v(T)\le V_M\,. \tag{3.5}$$

Similarly

$$v(0)\ge V_m\Rightarrow v(T)\ge V_m\,. \tag{3.5a}$$

Since $c(v)>0$ and $\phi(t)\le V_M$, and these functions are continuous, any solution of (3.1) which crosses the line $t=0$, will also cross the line $t=T$, and these solutions define a continuous map of the line $t=0$ into the line $t=\text{T}$. We have just observed that this map takes the closed interval on $t=0$,

$$V_m\le v\le V_M\,, \tag{3.6}$$

into the same closed interval on $t=T$.

Thus the differential equation defines a continuous map of the closed interval (3.6) of the real line into itself. By Brouwer's Fixed Point Theorem,[6] this map has a fixed point. In terms of the differential equation this means that there is a solution which crosses $t=0$ and $t=T$ at exactly the same value of v. Since the right member of (3.1) is periodic of period T and since (3.1) is an equation of the first order, this solution will be identical in any period strip t to $t+T$, i.e., this solution is periodic with period T.

(ii) *Uniqueness*

The differential equation (3.1) may be written as a system of differential equations

$$\frac{dv}{d\tau}=\phi-v$$

$$\frac{dt}{d\tau}=c(v)\,. \tag{3.7}$$

The parameter τ induces a flow in the t-v plane which is a continuous transformation of this plane. Thus a simply connected set in this plane will be transformed into another such set. The area of any set is altered by this flow. Indeed if we denote the area of a set as usual by

$$\iint_A dvdt\,, \tag{3.8}$$

we have

$$\frac{d}{d\tau}\iint_A dvdt=\iint_A Jdvdt\,, \tag{3.9}$$

where

$$J=\text{trace}\,\frac{\partial\left(\frac{dv}{d\tau},\frac{dt}{d\tau}\right)}{\partial(v,t)} \tag{3.10}$$

$$=\text{trace}\begin{vmatrix}-1 & \phi' \\ c' & 0\end{vmatrix}=-1\,.$$

Thus all areas are reduced by this flow (i.e., as τ increases).[7]

Suppose then that there exist two periodic solutions of (3.1), v_1 with period T_1 and v_2 with period T_2. If T_1 and T_2 are commensurable, there exists a pair of integers p and q, such that v_1 and v_2 have the period

$$T_3 = pT_1 + qT_2\,. \tag{3.11}$$

Consider the domain delimited by v_1 and v_2 as upper and/or lower boundaries and $t=0$ and $T=T_3$ as left and right boundaries. This domain must be congruent to the domain bounded by v_1 and v_2 and the abcissa $t=T_3$ and $t=2T_3$ since v_1 and v_2 have the period T_3. This, however, is impossible since the second domain is the image of the first under the flow induced by τ, and must therefore have a smaller area. Thus T_1 is a multiple[8] of T_2 and $v_1 \equiv v_2$, i.e., $T_1 = T_2 = T$.

The proof when T_1 and T_2 are incommensurable proceeds in a similar fashion.

(iii) *Stability*

Let $v_0(t)$ denote the periodic solution of (3.1) and suppose to the contrary that there exists a solution $v_1(t)$ of (3.1) with the property that

$$v_1(T) - v_0(T) = v_1(T) - v_0(0) \geq v_1(0) - v_0(0)\,. \tag{3.12}$$

Since $v_1(t)$ is not periodic, we must have a strict inequality in (3.12). Let $v_2(t)$ be a solution of (3.1), with the property

$$v_2(0) = v_1(T)\,. \tag{3.13}$$

In Fig. 3 the domain A is mapped under the flow into a domain congruent to the domain $A+B$. This is impossible since the area of B is positive. Thus

$$v_1(T) - v_0(T) < v_1(0) - v_0(0)\,. \tag{3.14}$$

Our assertion is proved when we observe that 0 and T in this argument can be replaced by $0+t$ and $T+t$ for any t, and that the area flow causes an exponential decrease in the area itself (viz., (3.8)-(3.10)). A significant consequence of this theorem is the following:

Corollary: If $\phi(t)$ is harmonic, the differential equation (3.1) can have no subharmonic nor any superharmonic solutions.

It is easy to see that this implies that no lumped circuit with a capacitance and a resistance can have subharmonic or superharmonic solutions.[9]

The above theorem can be extended to cases when $c(v) \not> 0$, but changes in sign. Of course when $c(v) < 0$, we may not have a physical situation. However, in trying to describe a device by endowing it with a nonlinear capacitance, the analytic expression for the capacitance may be valid and positive only for certain values of the voltage. When the voltage leaves this range, the expression for $c(v)$ may have no physical meaning. Nevertheless, the behavior of the device in the range of validity of the particular expression for capacity may be profitably studied by considering the fictitious behavior of the device outside of this range.

Rather than contrive theorems pertaining to this state of affairs, we consider a special case of interest to illustrate the possibilities.

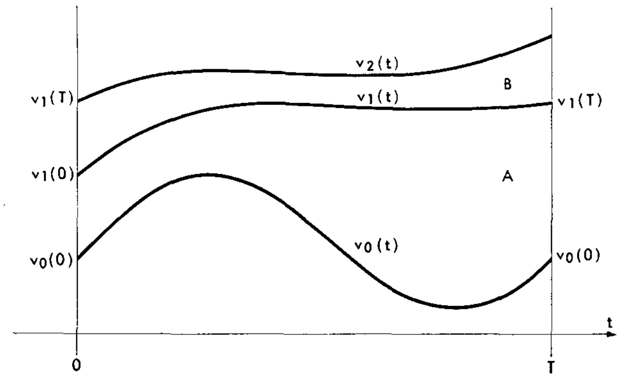


Figure 3

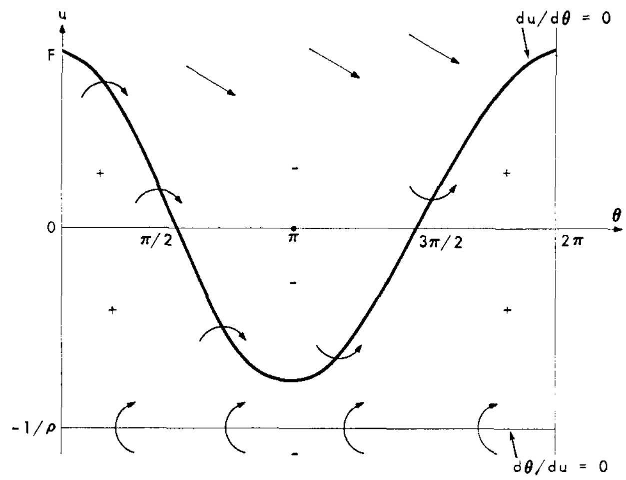


Figure 4

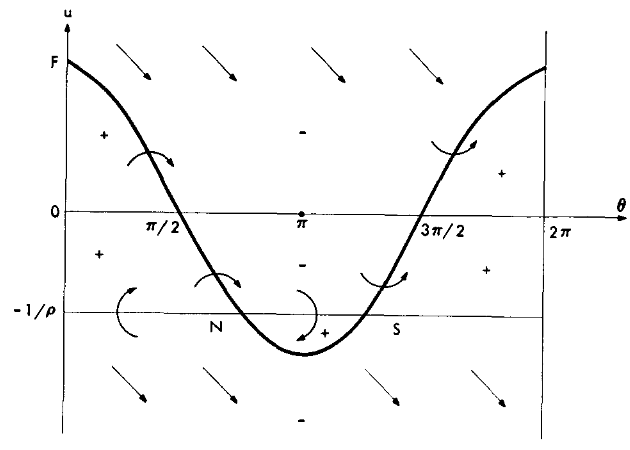


Figure 5

Suppose that

$$c(v) = c(1+\rho v)\,, \tag{3.15}$$

where c is a positive constant and ρ is a parameter considered to be small. When $\rho v < -1$, this expression for the capacitance of a device ceases to have physical meaning. For definiteness let us assume that

$$\phi(t) = F \cos \omega t, \qquad F > 0\,. \tag{3.16}$$

Inserting these expressions into (3.1) gives the differential equation

$$\frac{dv}{dt} = \frac{F \cos \omega t - v}{c(1+\rho v)}\,. \tag{3.17}$$

If we introduce the transformations

$$\theta = \omega t, \qquad v(t) = u(\theta), \qquad W = \omega c\,, \tag{3.18}$$

then (3.17) becomes

$$\frac{du}{d\theta} = \frac{F \cos \theta - u}{W(1+\rho u)}\,. \tag{3.19}$$

In Fig. 4 we plot signum $du/d\theta$ in a (θ, u) plane for the case that the parameter

$$\varepsilon = \rho F < 1\,. \tag{3.20}$$

Our previous theorem is applicable to this situation. We have only to observe that the fixed-point argument applies to the domain $|u| \leq F$ in this case, while all other arguments apply universally. We formulate these observations in the following:

- *Theorem*

The differential equation (3.17) has one and only one periodic solution. This solution has the frequency ω, it is stable and has the property that its amplitude is not greater than F.

Suppose now that the parameter $\varepsilon > 1$. Then the situation is as in Fig. 5.

The points S and N are singular points of the differential field, i.e., points when the numerator and denominator in (3.19) vanish simultaneously.

To determine the nature of these singular points, we proceed as follows. Write (3.19) as

$$\rho\frac{du}{d\rho} = \frac{(1+\varepsilon \cos \theta) - (1+\rho u)}{W(1+\rho u)}\,. \tag{3.21}$$

Let

$$1+\rho u = y\,, \qquad \rho du = dy$$

$$1+\varepsilon \cos \theta = x\,, \qquad d\theta = \frac{dx}{-\varepsilon \sin \theta} = \frac{dx}{\mp\sqrt{\varepsilon^2-(x-1)^2}}\,. \tag{3.22}$$

Here $x=0$, $y=0$ are the singular points. The upper sign refers to a neighborhood of N and the lower sign to a neighborhood of S.

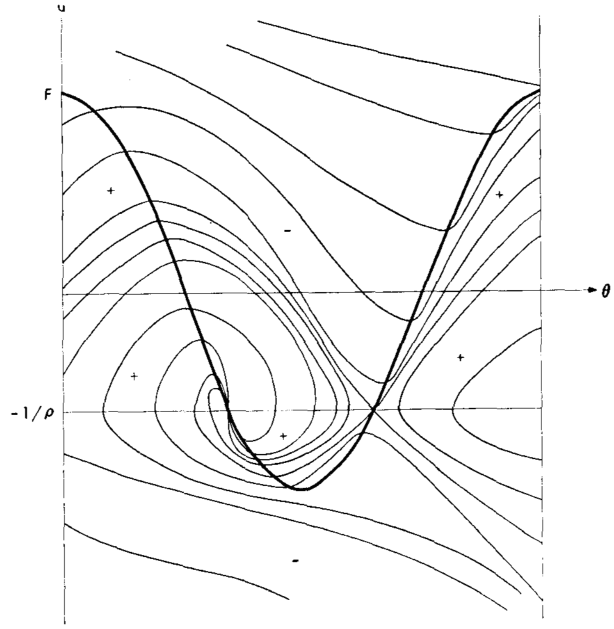


Figure 6

The differential equation is now

$$\frac{dy}{dx} = \frac{(x-y)}{\mp\sqrt{\varepsilon^2-(x-1)^2}\,Wy}\,. \tag{3.23}$$

Now we reduce (3.23) to a neighborhood of S and N and obtain

$$\frac{dy}{dx} = \frac{x-y}{\mp\sqrt{\varepsilon^2-1}\,Wy}\,. \tag{3.24}$$

Now consider

$$\begin{aligned}\Delta &= 1 \mp 4\sqrt{\varepsilon^2-1}\,W \\ D &= \mp\sqrt{\varepsilon^2-1}\,W\end{aligned}\,. \tag{3.25}$$

For S we have $\Delta > 0$ and $D > 0$ so that S is a saddle point.[10] For N we have $D < 0$ and

$$\Delta > 0 \text{ if } \frac{1}{4\sqrt{\varepsilon^2-1}} > W, \qquad \text{(low frequency)} \tag{3.25a}$$

$$\Delta = 0 \text{ if } \frac{1}{4\sqrt{\varepsilon^2-1}} = W \tag{3.25b}$$

$$\Delta < 0 \text{ if } \frac{1}{4\sqrt{\varepsilon^2-1}} < W \qquad \text{(high frequency)}\,. \tag{3.25c}$$

In the first case N is a node, in the second case it is a degenerate (or spiralar) node, and in the third case it is a spiral. We remark that the primary significance of the occurrence of S and N is that they are points where solutions of the differential equation cross. Indeed at S two solutions cross while at N infinitely many cross.

In Fig. 6 we sketch a possible state of affairs for the case of low frequency. This Figure illustrates a number of particular solutions of the differential equation.

The arguments concerning existence, uniqueness and stability of a periodic solution which we have discussed previously break down in this case ($\varepsilon>1$). For one thing, the mapping is no longer into, because the direction field points down both above and below the θ-axis (or t-axis). Because of singular points the map is no longer continuous.

To study the properties of periodic solutions in this case requires different procedures. This question forms the contents of the following section. The methods which we use are topological also, but they are conducted in function spaces rather than in Euclidean spaces. While the methods to be used are more complicated than those already considered, they motivate and justify a perturbational procedure which obtains concrete analytic information concerning the periodic solutions.

• 4. *The properties of periodic solution when there are singular points*

In this section we will consider the differential equation

$$\frac{dv}{dt}=\frac{F\cos\omega t-v}{c(1+\rho v)}, \tag{4.1}$$

in the case that there are singular points, i.e., when

$$\varepsilon=\rho F>1. \tag{4.2}$$

We will prove the following

• *Theorem*

There exists a periodic solution of period $2\pi/\omega$ of the differential equation. This solution is an analytic function of ρ and its amplitude is less than $1/\rho$.

This theorem assures us that there is a periodic solution and that it lies entirely away from the singular points discussed in the previous section. Of course the methods used here apply without exception to the case when there are no singular points. With this information we are able to prove the uniqueness and stability of this solution. We state this as a theorem.

• *Theorem*

The periodic solution of (4.1) with amplitude less than $1/\rho$ is unique among all such solutions with this property and stable in the sense that neighboring solutions tend to it as $t\to\infty$.

• *Proof*

(i) *Uniqueness*

Since we are confining our attention to periodic solutions which are bounded away from the singular points, our proof of uniqueness in Section 3 is valid here. We present instead, however, an alternate proof because of a property of the periodic solutions which it derives and makes use of. Let

$$q(v)=\int_0^v c(\sigma)d\sigma. \tag{4.3}$$

($q(v)$ is of course, the charge on the capacitor. However, we need not be concerned with this fact.) Now if v is a periodic function of time, so then is $q(v)$. Then dq/dt is also a periodic function of time. Let the period in question be denoted by $T=2\pi/\omega$. Thus we have

$$\int_0^T\frac{dq}{dt}\,dt=q(T)-q(0)=0. \tag{4.4}$$

Thus dq/dt has mean zero over a period. But (4.1) may be written as

$$c(v)\frac{dv}{dt}-F\cos\omega t=-v \tag{4.5}$$

or

$$\frac{dq}{dt}-F\cos\omega t=-v. \tag{4.6}$$

The left member has mean zero over a period. Thus v itself has this property.[11]

Now suppose there were two solutions of period T, both of which have amplitudes less than $1/\rho$. If this is the case these two periodic solutions can not cross since they are bounded away from the singular points, and no two different solutions of a first-order differential equation can cross at a regular point of the trajectory field. But both of these periodic solutions must have mean zero. Thus they must cross. This contradiction implies the result.

(ii) *Stability*

This fact follows exactly as in the stability proof of Section 3 when we use the observation that the periodic solution is bounded away from the singular points. We must note, however, that in the case of Section 3, i.e., no singular points, every solution tended toward the periodic solution. In the present case only neighboring solutions need do this.

We now turn to the proof of the existence theorem. It proceeds in two parts. The first part assumes that there are solutions of (4.1) which start out at $t=0$ and cross the line $t=T(=2\pi/\omega)$ and which depend analytically on ρ. If we examine the phase plane diagram sketched in Fig. 6, we see that there are trajectories of this type. Using this assumption it is shown that there is a value of $v(0)=A$ such that the solution with this value at $t=0$ has the property $v(T)=A$. This solution is then of course periodic, since our equation is of the first order and its coefficients have period T. The second part of the proof produces a class of solutions which are analytic in ρ and which cross both $t=0$ and $t=T$. Moreover, it shows that the solution with $v(0)=A$ lies in this class.

The first part of the proof produces the result that $v(0)=A$ is a quantity which depends on the frequency ω. This is to be expected since at each applied frequency only certain trajectories will be periodic. This fact also forms the basis of the perturbational procedure to be explored in Section 5. This relation, of course, justifies the use of this procedure.

The second part of the proof requires that the applied frequency be not too small.

We now turn to the

Proof of Part 1

The differential equation (4.1) may be written in the following form

$$c\left(v+\frac{\rho}{2}v^2\right)_t=F\cos\omega t-v\,. \tag{4.7}$$

Equation (4.7) is equivalent to the integral equation

$$v=\frac{1}{c}\left[-\frac{\rho}{2}cv^2-\int_0^t v(\tau)d\tau+\frac{F}{\omega}\sin\omega t+c\left(A+\frac{\rho}{2}A^2\right)\right], \tag{4.8}$$

where we have assigned

$$v(0)=A\,. \tag{4.9}$$

Now we assume that v depends analytically on ρ for ρ near zero, and we denote this dependence on ρ by affixing a subscript ρ to various quantities. Now suppose that v_ρ had a period T_ρ. Then $v_\rho(T_\rho)=v(0)$, or

$$v_\rho(T_\rho)-A=0\,. \tag{4.10}$$

(4.10) is an equation for T_ρ in terms of A. Now indeed T_ρ is known, for we want to produce a solution of period $T=2\pi/\omega$. Thus while we regard (4.10) as an equation for T_ρ in terms of A, it is really an equation for A in terms of T_ρ. Inserting v_ρ as given in (4.8) into (4.10) yields

$$0=\int_0^{T_\rho}v_\rho(\tau)d\tau-\frac{F}{\omega}\sin\omega T_\rho\equiv P(v_\rho,T_\rho,A) \tag{4.11}$$

for the periodicity condition. We want to solve this equation for T_ρ in terms of A for ρ near zero. Since v_ρ depends analytically on ρ for ρ near zero, so will T_ρ. Thus we apply the implicit function theorem to (4.11), i.e., we show that we can solve (4.11) for T_ρ in terms of A for $\rho=0$. Then we show that the derivative of T_ρ with respect to A at $\rho=0$ does not itself vanish at the solution corresponding to $\rho=0$. This is sufficient since all quantities are analytic in ρ. At $\rho=0$, (4.11) becomes

$$0=\int_0^{T_0}v_0(\tau)d\tau-\frac{F}{\omega}\sin\omega T_0\,. \tag{4.12}$$

From (4.7), we compute $v_0(t)$, viz.,

$$v_0(t)=A(\cos\omega t+\sin\omega t) \tag{4.13}$$

provided

$$A=\frac{F}{1+c^2\omega^2}\,. \tag{4.14}$$

Thus

$$T_0=2\sqrt{\frac{F}{F-A}}\,. \tag{4.15}$$

Now we want to find

$$\left.\frac{dT}{dA}\right|_{\rho=0}=\frac{dP_0/dA}{dP_0/dT_0}\,. \tag{4.16}$$

However, when we insert (4.13) and (4.15) into (4.12) (i.e., set $\rho=0$ in 4.12), we find that (4.12) is satisfied identically and there is nothing left to differentiate (i.e., $P_0(v_0,T_0,A)\equiv0$). Thus before we let $\rho\to0$ in (4.12) we divide P_ρ by ρ. This does not affect our previous arguments. Denote

$$Q_\rho=P_\rho/\rho \tag{4.17}$$

and let

$$T_\rho=T_0+\rho\eta(\rho)\,,\qquad \eta(0)\neq0\,, \tag{4.18}$$

with T_0 as given by (4.15). (If $\eta(0)=0$, we must divide P_ρ by a higher power of ρ in (4.17).) Now we find by a simple computation that

$$\lim_{\rho\to0}Q(v_\rho,T_0+\rho\eta(\rho),A)=0 \tag{4.19}$$

becomes the equation

$$\eta(0)(A-F)=0\,. \tag{4.20}$$

Then

$$\frac{d\eta(0)}{dA}=-\frac{dQ_0/dA}{dQ_0/d\eta(0)}=-\frac{\eta(0)}{A-F}\,. \tag{4.21}$$

Thus we may indeed find T_ρ in terms of A for ρ near zero.

We now turn to the

Proof of Part 2

We find it more convenient to use the (θ,u) variables introduced in Section 3. ($\omega t=\theta$, $v(t)=u(\theta)$, $\omega c=W$). Then our equation may be written in the form

$$W\left(u+\frac{\rho}{2}u^2\right)_\theta=-u+F\cos\theta,\ u(0)=A\,. \tag{4.22}$$

or

$$u=\frac{1}{W}\left[-\frac{\rho}{2}Wu^2+F\sin\theta-\int_0^\theta u\,d\phi+W\left(A+\frac{\rho}{2}A^2\right)\right]. \tag{4.23}$$

We consider (4.23) to be a map defined on the Banach space of functions of two variables, $u(\theta,\rho)$. The elements of our Banach space B are to be continuous in θ and in ρ and analytic in ρ. θ varies in an interval

$$0\leq\theta\leq l\,, \tag{4.24}$$

where $l>2\pi$. ρ varies in

$$|\rho|\leq R\,, \tag{4.25}$$

where R is to be specified. Of course u as a function of ρ is continuous for ρ on the set $|\rho|\leq R$ but analytic on the set $\rho<R$. The norm of u is

$$||u||=\max_{\substack{0\le\theta\le l\\ |\rho|\le R}}|u(\theta,\rho)|\,. \tag{4.26}$$

Let S be the set in B, where

$$||u||\le M, \tag{4.27}$$

M to be specified, and

$$u(0)=A\,. \tag{4.28}$$

We see that S is a closed set in B. We now wish to determine for which A and W the map (4.23) has a fixed point in S.

Let $\bar{u}_j$ be the image of u_j under (4.23). Then

$$\bar{u}_1-\bar{u}_2=-\frac{\rho}{2}(u_1+u_2)(u_1-u_2)-\frac{1}{W}\int_0^\theta(u_1-u_2)d\theta\,. \tag{4.29}$$

Then

$$||\bar{u}_1-\bar{u}_2||\le\left(\frac{l}{W}+RM\right)||u_1-u_2||\,. \tag{4.30}$$

Thus if

$$\frac{l}{W}+RM<1\,, \tag{4.31}$$

the map (4.23) will be contracting on S. This will be the case if

$$W>l \tag{4.32}$$

and whatever M is, R is chosen to be sufficiently small.

Now $\bar{u}$ is continuous in θ and analytic in ρ whenever u is, and indeed $\bar{u}(0)=A$ if $u(0)=A$. Thus the map takes B into itself. To see if the map takes S into itself, we make the following estimate from (4.23):

$$||\bar{u}||\le\frac{R}{2}M^2+\frac{F}{W}+\frac{lM}{W}+|A|+\frac{R}{2}|A|^2\,. \tag{4.33}$$

Thus since we must have $||\bar{u}||\le||u||\le M$, we need consider the inequality

$$\frac{R}{2}M^2+\frac{F}{W}+\frac{lM}{W}+|A|+\frac{R}{2}|A|^2\le M\,, \tag{4.34}$$

when $|A|\le M$.

This is only possible if the following polynomial in M has a positive root:

$$\frac{R}{2}M^2+\left(\frac{l}{W}-1\right)M+\frac{F}{W}+|A|+\frac{R}{2}|A|^2=0\,. \tag{4.35}$$

The roots of this polynomial are

$$M=\frac{1}{R}\left[1-\frac{l}{W}\pm\sqrt{\left(1-\frac{l}{W}\right)^2-2R\left(\frac{F}{W}+|A|+\frac{R}{2}|A|^2\right)}\right]. \tag{4.36}$$

If these roots are real, they are both positive, if one is. For them to be positive we must have

$$\frac{l}{W}<1\,. \tag{4.37}$$

For the roots to be real, we must have

$$\left(1-\frac{l}{W}\right)^2-2R\left(\frac{F}{W}+|A|+\frac{R}{2}|A|^2\right)\ge0\,. \tag{4.38}$$

This is a condition on A. In addition to $|A|\le M$, A must be smaller than the larger root in (4.36). This gives a second condition on A:

$$|A|\le\frac{1}{R}\left[1-\frac{l}{W}+\sqrt{\left(1-\frac{l}{W}\right)^2-2R\left(\frac{F}{W}+|A|+\frac{R}{2}|A|^2\right)}\right]. \tag{4.39}$$

or

$$R|A|-\left(1-\frac{l}{W}\right)\le\sqrt{\left(1-\frac{l}{W}\right)^2-2R\left(\frac{F}{W}+|A|+\frac{R}{2}|A|^2\right)}\,. \tag{4.40}$$

The left member of (4.40) is negative or not. If it is negative, the only restriction on A is (4.38). If it is positive we square (4.40) to obtain

$$R^2|A|^2-2R\left(1-\frac{l}{W}\right)|A|\le-2R\left(\frac{F}{W}+|A|+\frac{R}{2}|A|^2\right), \tag{4.41}$$

or

$$\left(R^2+\frac{R^2}{W}\right)|A|^2+2R|A|+\frac{2R}{W}F\le0\,. \tag{4.42}$$

Equation (4.42) is never satisfied for any A. Thus the left member of (4.40) must be negative, i.e.,

$$|A|\le\frac{W-l}{RW}\,. \tag{4.43}$$

Since $W>l$, this inequality is possible. Now (4.38) asserts that

$$R^2|A|^2+2R|A|+\frac{2RF}{W}-\left(1-\frac{l}{W}\right)^2\le0\,. \tag{4.44}$$

For this to be satisfied, the corresponding polynomial in $|A|$ must have a positive root. Its roots are

$$|A|=\frac{1}{2R^2}\left[-2R\pm\sqrt{4R^2-4R^2\left(\frac{2RF}{W}-\left(1-\frac{l}{W}\right)^2\right)}\right]. \tag{4.45}$$

For the larger of these two numbers to be positive, we must have,

$$-4R^2\left(\frac{2RF}{W}-\left(1-\frac{l}{W}\right)^2\right)\geq 0\,. \tag{4.46}$$

or

$$2RF\leq W\left(1-\frac{l}{W}\right)^2. \tag{4.47}$$

Now for any fixed $W>l$, (4.47) may be arranged by choosing R sufficiently small. Or for a fixed $RF(\sim\varepsilon>1)$, (4.47) may be arranged by choosing W sufficiently large.

There remains only to see if the A produced in the proof of Part 1 of this theorem satisfies the restriction (4.43).

From (4.14) we have

$$A=\frac{F}{1+W^2}\,. \tag{4.48}$$

Thus by making W sufficiently large we may make A as small as we please. Combining (4.43) and (4.45) we get

$$\frac{F}{1+W^2}\leq\frac{W-l}{RW} \tag{4.49}$$

or

$$RF\leq\frac{1+W^2}{W}\,(W-l)\,. \tag{4.50}$$

Thus for a given $RF(\sim\varepsilon>1)$, by making W sufficiently large, (4.50) may be arranged.

Equation (4.36) shows that

$$|A|\leq M<\frac{1}{R}\leq\frac{1}{|\rho|}\,. \tag{4.51}$$

Thus our periodic solution is bounded in amplitude by $1/|\rho|$, i.e., is bounded away from the singular points.

This concludes our proof.

III. Perturbational analysis of the differential equation

In this part of the paper we will apply a perturbational procedure to the differential equation to compute approximations to the periodic solutions. In Section 5 we will consider harmonic response to harmonic input, while in Section 6 we consider combination oscillations to so-called carrier and signal inputs. In Section 6 the phenomena of amplification and conversion are discussed.

5. *Harmonic response to harmonic input*

In this section we will consider the equation

$$c(1+\rho v)v_t=-v+\phi \tag{5.1}$$

when

$$\phi(t)=F\cos\omega t\,, \tag{5.2}$$

i.e., the line is being driven harmonically. We will investigate the periodic solutions of (5.1).

First let

$$\theta=\omega t\,,\qquad u(\theta)=v(t)\,,\qquad W=\omega c\,. \tag{5.3}$$

Then our equation is

$$W(1+\rho u)u_\theta=-u+F\cos\theta\,. \tag{5.4}$$

For definiteness, let us suppose that

$$u(0)=A\,. \tag{5.5}$$

Now in fact A is not known, while W is indeed prescribed in advance. However, we have seen in Section 4, that this reversing procedure will enable one to derive the amplitude-versus-frequency relation for a periodic $u(\theta)$. This is merely a point of view and only serves to facilitate the procedure. Thus let us look for u and W which satisfy (5.4) subject to (5.5) and a condition of periodicity on u, in the form of perturbation series in the parameter ρ, viz:

$$u=u_0+\rho u_1+\ldots\,, \tag{5.6}$$

$$W=W_0+\rho W_1+\ldots\quad. \tag{5.7}$$

We cause (5.6) to satisfy (5.5) by requiring that

$$u_0(0)=A \tag{5.8}$$

$$u_i(0)=0\,,\qquad i>0\,. \tag{5.9}$$

Now if we insert (5.6) and (5.7) into (5.4), and equate the coefficients of the powers of ρ to zero, we obtain a sequence of equations, the first two members of which are

$$W_0u_0'+u_0=F\cos\theta \tag{5.10}$$

$$W_0u_1'+u_1=-W_0u_0u_0'-W_1u_0'\,. \tag{5.11}$$

Here, and in what follows, a prime denotes differentiation with respect to θ. Now the most general solution of (5.10) is

$$u_0=\lambda\exp\left(\frac{\theta}{W_0}\right)+\alpha_0\sin\theta+\beta_0\cos\theta\,. \tag{5.12}$$

Since u_0 is to be periodic because u is, and W_0 is to be real because W is, we must have $\lambda=0$. If we insert (5.12) into (5.10), and set the coefficients of $\sin\theta$ and $\cos\theta$ to zero, we obtain two equations for the determination of α_0, β_0, and W_0.

These equations are

$$\alpha_0-W_0\beta_0=0 \tag{5.13}$$

$$W_0\alpha_0+\beta_0=F\,. \tag{5.14}$$

Thus

$$\alpha_0=\frac{W_0F}{1+W_0^2} \tag{5.15}$$

and

$$\beta_0=\frac{F}{1+W_0^2}\,. \tag{5.16}$$

Now if we apply (5.8) to (5.12), we have $A=\beta_0$ or, using (5.16),

$$W_0{}^2=\frac{F-A}{A}. \tag{5.17}$$

If we introduce the notation

$$\mu=\frac{A}{F}, \tag{5.18}$$

equation (5.17) becomes

$$W_0{}^2=\frac{1-\mu}{\mu}. \tag{5.19}$$

From this we may conclude that $\mu\leq 1$.

Now if we insert (5.12) into (5.11), (5.11) becomes

$$\begin{aligned} W_0u_1'+u_1=&-W_0[\tfrac{1}{2}(\alpha_0{}^2-\beta_0{}^2)\sin 2\theta+\alpha_0\beta_0\cos 2\theta] \\ &-W_1(\alpha_0\cos\theta-\beta_0\sin\theta). \end{aligned} \tag{5.20}$$

The most general periodic solution of (5.20) is

$$u_1=\alpha_{11}\sin\theta+\beta_{11}\cos\theta+\alpha_{12}\sin 2\theta+\beta_{12}\cos 2\theta, \tag{5.21}$$

where α_{11}, β_{11}, α_{12}, and β_{12} are determined by inserting (5.21) into (5.20) and equating the coefficients of $\sin\theta$, $\cos\theta$, $\sin 2\theta$, and $\cos 2\theta$ to zero. This procedure leads to the following four equations for determining α_{11}, β_{11}, α_{12}, β_{12}, and W_1.

$$\begin{aligned} &\alpha_{11}-W_0\beta_{11}=W_1\beta_0 \\ &W_0\alpha_{11}+\beta_{11}=-W_1\alpha_0 \\ &\alpha_{12}-2W_0\beta_{12}=-\tfrac{1}{2}W_0(\alpha_0{}^2-\beta_0{}^2) \\ &2W_0\alpha_{12}+\beta_{12}=-W_0\alpha_0\beta_0. \end{aligned} \tag{5.22}$$

These four linear equations yield

$$\alpha_{11}=W_1F\frac{1-W_0{}^2}{(1+W_0{}^2)^2} \tag{5.23}$$

$$\beta_{11}=-\frac{2W_0W_1F}{(1+W_0{}^2)^2} \tag{5.24}$$

$$\alpha_{12}=-\tfrac{1}{2}\frac{F^2W_0[5W_0{}^2-1]}{(1+4W_0{}^2)(1+W_0{}^2)^2} \tag{5.25}$$

$$\beta_{12}=\frac{F^2W_0{}^2(W_0{}^2-2)}{(1+4W_0{}^2)(1+W_0{}^2)^2}. \tag{5.26}$$

Now if we apply (5.9) to (5.21), we obtain $\beta_{11}+\beta_{12}=0$, or from (5.24) and (5.26)

$$-2W_0W_1F+\frac{W_0{}^2F^2(W_0{}^2-2)}{1+4W_0{}^2}=0. \tag{5.27}$$

This equation may be solved for W_1 in terms of W_0 and F, viz.,

$$W_1=\frac{W_0F}{2}\,\frac{W_0{}^2-2}{4W_0{}^2+1}. \tag{5.28}$$

Combining this and (5.7) gives us the following expression for W

$$W=W_0\left[1+\frac{\rho F}{2}\,\frac{W_0{}^2-2}{4W_0{}^2+1}\right]+0(\rho^2). \tag{5.29}$$

Let us introduce the abbreviation

$$\varepsilon=\rho F, \tag{5.30}$$

in (5.19) and (5.29). This gives finally

$$W=\sqrt{\frac{1-\mu}{\mu}}\left[1+\frac{\varepsilon}{2}\,\frac{1-3\mu}{4-3\mu}\right] \tag{5.31}$$

to within an error of order ρ^2. Introduction of (5.19) into (5.15), (5.16), (5.23)–(5.26) gives us α_0, β_0, α_{11}, β_{11}, α_{12} and β_{12} in terms of A, F, and μ. These expressions are

$$\begin{aligned} &\alpha_0=-A\frac{1-\mu}{\mu} \\ &\beta_0=A, \\ &\alpha_{11}=A\frac{1-\mu}{\mu}\,\frac{1-3\mu}{4-3\mu}(2\mu-1), \\ &\beta_{11}=-2AF(1-\mu)\frac{1-3\mu}{4-3\mu}, \\ &\alpha_{12}=-\tfrac{1}{2}A^2\frac{1-\mu}{\mu}\,\frac{5-6\mu}{4-3\mu} \\ &\beta_{12}=AF\frac{(1-\mu)(1-3\mu)}{4-3\mu}. \end{aligned} \tag{5.32}$$

Inserting (5.32) into (5.12) and (5.21) and the result into (5.6), we obtain u in terms of A, F, μ, and ε;

$$\begin{aligned} u=&-\sqrt{\frac{1-\mu}{\mu}}\left[-1\frac{\varepsilon}{F}\,\frac{1-3\mu}{4-3\mu}(2\mu-1)\right]\sin\theta \\ &+A\left[1-2\varepsilon(1-\mu)\frac{1-3\mu}{4-3\mu}\right]\cos\theta \\ &-\tfrac{1}{2}A\mu\varepsilon\sqrt{\frac{1-\mu}{\mu}}\,\frac{5-6\mu}{4-3\mu}\sin 2\theta \\ &+A\varepsilon\frac{(1-\mu)(1-3\mu)}{4-3\mu}\cos 2\theta \end{aligned} \tag{5.33}$$

to within terms of order ρ^2. From (5.33) we may compute the ratio of the output amplitudes squared to the input amplitude squared. Using Parseval's equality, we obtain

$$\frac{||u||^2}{F^2}=\mu\left[1-2(1-\mu)\frac{1-3\mu}{4-3\mu}\varepsilon\right], \tag{5.34}$$

to within terms of order ρ^2.

There remains, then, to analyze the response relation (5.31).

(i) *Analysis of response curve*

In Fig. 7 we plot the curve (5.31) for several values

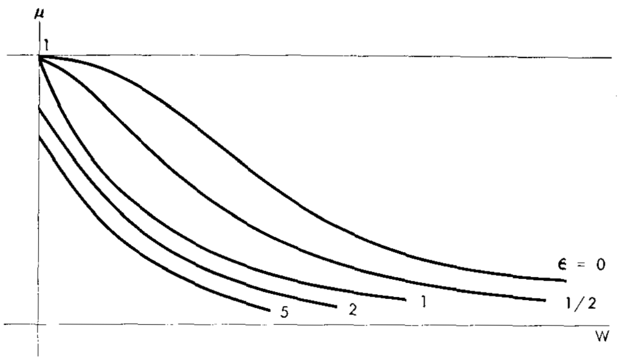


Figure 7

of ε. This is a plot of the normalized amplitude at $\theta=0$, A/μ, of the periodic solution versus frequency. All curves are asymptotic to the W axis.

For $\varepsilon=0$ the response curve is a bell-shaped curve tangent to $\mu=1$ at $W=0$. As ε increases from zero to one the same situation prevails, the bells becoming lower. This situation corresponds to the fact pointed out in Part II, that the amplitude of the periodic solution lies somewhere between 0 and F or in terms of μ, for μ between zero and one. We see that this is indeed the case and that at least for small frequency the amplitude at $\theta=0$, i.e., A, gets very near F. Of course for larger frequency the amplitude at zero is less than F, but it might get as large as F for $\theta\neq 0$. This is probably what happens and could be decided by analyzing (5.34). When $\varepsilon>1$ the response curve no longer passes through $\mu=1$. The response curves rise from zero at $W=\infty$ to

$$\mu=\tfrac{1}{3}\,\frac{8+\varepsilon}{2+\varepsilon} \tag{5.35}$$

at $\mu=0$. This is consistent with what we have seen in Part II. Namely, that the amplitude of the solution lies between zero and F if $\varepsilon<1$. But if $\varepsilon>1$, the amplitude is less than F and only possibly as large as $1/\rho$. The intercept given in (5.35) decreases from one at $\varepsilon=1$ to 2/3 at $\varepsilon=\infty$.

The actual curve of normalized amplitude versus normalized amplitude at $\theta=0$, (5.34), is plotted in Fig. 8. The curves for $\varepsilon>1$ do not cover the entire range $0\le\mu\le 1$. The curves for $0<\varepsilon\le 1$ lie slightly above $||u||^2/F^2=1$ for μ slightly less than one. This is probably due to a truncation error.

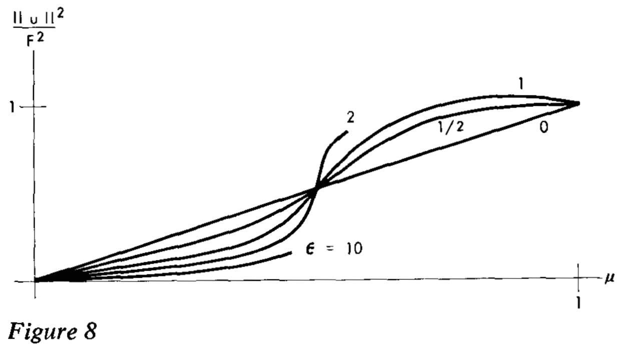


Figure 8

(ii) *Subharmonics*

In this section we interpolate a short plausibility argument to demonstrate the nonexistence of subharmonic response based upon simple perturbational methods. This fact was rigorously proved in Section 3, however, by using not too simple methods. We consider the first perturbation equation, viz.,

$$W_0u_0'+u_0=F\cos\theta\ . \tag{5.10}$$

If there were a subharmonic response, this equation should, e.g., have a solution of the form

$$u_0=\alpha_0\sin\frac{\theta}{n}+\beta_0\cos\frac{\theta}{n}\,. \tag{5.36}$$

Then we must have

$$-W_0\frac{\beta_0}{n}+\alpha_0=0\ , \tag{5.37}$$

and

$$W_0\frac{\alpha_0}{n}+\beta_0=0 \tag{5.38}$$

for the determination of α_0 and β_0. To find a nonzero solution to (5.37) and (5.38), the determinant of the coefficient matrix must vanish, i.e.,

$$1+\frac{W_0^2}{n^2}=0\ . \tag{5.39}$$

Clearly no value of n or real W_0 will cause this result. From a mathematical point of view the reason for the nonexistence of subharmonic response to the differential equation is because the order of the equation is too low, it being only a first-order equation, i.e., the equation, being too simple, does not possess many solutions and in particular, does not possess subharmonic solutions. It is fairly easy to see that if the equation were of second order that there would be subharmonic solutions. By coupling the capacitor with an inductance, the order of the equation will be raised and subharmonic response will exist. These matters will be discussed in a subsequent paper.

• 6. *Amplification and conversion*

In this section we will consider the response of our differential equation, when it is driven by a pair of harmonic functions of different frequencies, in particular

$$\phi(t)=F_1\cos\omega_1t+F_2\cos\omega_2t\ . \tag{6.1}$$

We will see below that a solution to our differential equation in response to this forcing term will have frequency components in the frequencies ω_1, ω_2, $\omega_1+\omega_2$, $\omega_1-\omega_2$ and many others. Our point of view in this section will be slightly different from that of the previous section. This is true because more natural and pertinent questions are presented for (6.1). These questions concern the phenomena of amplification and conversion. To establish the ideas, suppose that F_1 is small compared to

F_2. The response to $\phi(t)$ in the frequency ω_1 will have an amplitude which depends on F_1 and because of the presence of $F_2 \cos \omega_2 t$ this amplitude may depend on F_2, i.e., the response in frequency ω_1 borrows energy from the input of frequency of ω_2. Were this to be the case, we could then consider the device to be an amplifier in the specific frequency ω_1. Similarly the fact that the response has components of frequencies other than ω_1 and ω_2, in particular $\omega_1 \pm \omega_2$ shows that energy is converted from the frequencies ω_1 and ω_2 into other frequencies. This phenomenon of conversion can also be exploited as amplification.

To consider what happens we proceed as follows. Consider the differential equation

$$c(1+\rho v)v_t = -v + F_1 \cos \omega_1 t + F_2 \cos \omega_2 t\,. \tag{6.2}$$

The perturbation procedure applied to this equation proceeds in a straightforward manner. However, it is not clear how to apply initial conditions because of the presence of multiple frequencies. We must, however, pursue the perturbation procedure in order to motivate the method of undetermined coefficients which we will use.

We suppose then for the time being that

$$v = v_0 + \rho v_1 + \ldots \tag{6.3}$$

Insertion of (6.3) into (6.2) and equating the coefficients of the various power of ρ to zero gives rise to a system of equations, the first two of whose members are

$$cv_0' + v_0 = F_1 \cos \omega_1 t F_2 \cos \omega_2 t \tag{6.4}$$

$$cv_1' + v_1 = -cv_0 v_0'\,. \tag{6.5}$$

The solution of (6.4) is

$$v_0 = \alpha_{10} \sin \omega_1 t + \beta_{10} \cos \omega_1 t + \alpha_{20} \sin \omega_2 t + \beta_{20} \cos \omega_2 t\,, \tag{6.6}$$

with α_{10}, β_{10}, α_{20}, and β_{20} determined in the usual manner.

With v_0 determined, (6.5) becomes

$$\begin{aligned} cv_1' + v_1 = {} & F_1\omega_1 \sin \omega_1 t + F_2\omega_2 \sin \omega_2 t \\ & + M_1 \sin 2\omega_1 t + N_1 \cos 2\omega_1 t \\ & + M_2 \sin 2\omega_2 t + N_2 \cos 2\omega_2 t \\ & + M_3 \sin(\omega_1+\omega_2)t + N_3 \cos(\omega_1+\omega_2)t \\ & + M_4 \sin(\omega_1-\omega_2)t + N_4 \cos(\omega_1-\omega_2)t\,. \end{aligned} \tag{6.7}$$

Here $M_1, M_2, \ldots, N_4$ are certain constants.

The solution of (6.7) is a sum of sines and cosines of frequencies ω_1, ω_2, $2\omega_1$, $2\omega_2$, and $\omega_1 \pm \omega_2$. We use this procedure to note that if we look for a solution of (6.2) in the form of sines and cosines of ω_1, ω_2, $2\omega_1$, $2\omega_2$, $\omega_1+\omega_2$, and $\omega_1-\omega_2$ that the coefficients of the terms of frequencies ω_1 and ω_2 are of order ρ^0, the coefficients of the terms of frequencies $2\omega_1$, $2\omega_2$, $\omega_1+\omega_2$ and $\omega_1-\omega_2$ are of order ρ^1, and by continuing this procedure, the coefficients of terms of other frequencies (e.g., $3\omega_1$, $3\omega_2$, $2(\omega_1+\omega_2)$, $2(\omega_1-\omega_2)$, $2\omega_1 \pm \omega_2$, $\omega_1 \pm \omega_2$, etc.) are of higher order in ρ. With this information we may proceed with our analysis as follows:

We look for a solution of (6.2) in the form

$$\begin{aligned} v = {} & \alpha_{11} \sin \omega_1 t + \beta_{11} \cos \omega_1 t + \alpha_{12} \sin \omega_2 t + \beta_{12} \cos \omega_2 t \\ & + \alpha_{21} \sin 2\omega_1 t + \beta_{21} \cos 2\omega_1 t + \alpha_{22} \sin 2\omega_2 t + \beta_{22} \cos 2\omega_2 t \\ & + \alpha_{31} \sin(\omega_1+\omega_2)t + \beta_{31} \cos(\omega_1+\omega_2)t + \alpha_{32} \sin(\omega_1-\omega_2)t \\ & + \beta_{32} \cos(\omega_1-\omega_2)t \\ & + \ldots \end{aligned} \tag{6.8}$$

We plan to insert this expression into (6.2). From what we have just observed from the perturbation procedure concerning orders of magnitude in ρ, we see that we may neglect any term which arises which is not of one of the six frequencies displayed in (6.8). The details of this enterprise are not noted. We merely remark that after the steps are carried out, we obtain an expression involving the 12 functions, $\sin \omega_1 t, \ldots, \cos(\omega_1-\omega_2)t$. Equating the coefficients of these 12 functions to zero gives us 12 equations, coupled only in pairs, for the determination of the 12 constants, $\alpha_{11}, \ldots, \beta_{32}$. If we introduce the abbreviations

$$W_1 = \omega_1 c \text{ and } W_2 = \omega_2 c\,, \tag{6.9}$$

these 12 equations are:

$$\sin \omega_1 t[\alpha_{11} - W_1\beta_{11}] = 0 \tag{6.10}$$

$$\cos \omega_1 t[-W_1\alpha_{11} - \beta_{11}] = -F_1 \tag{6.11}$$

$$\sin \omega_2 t[\alpha_{12} - W_2\beta_{12}] = 0 \tag{6.12}$$

$$\cos \omega_2 t[-W_2\alpha_{12} - \beta_{12}] = -F_2 \tag{6.13}$$

$$\sin 2\omega_1 t\left[\alpha_{21} - 2W_1\beta_{21} - 2W_1\beta_{11}{}^2 - \frac{\rho}{2} W_1(\alpha_{11}{}^2 + \beta_{11}{}^2)\right] = 0\,. \tag{6.14}$$

$$\cos 2\omega_1 t[-2W_1\alpha_{21} - \beta_{21} - \rho W_1\alpha_{11}\beta_{11}] = 0 \tag{6.15}$$

$$\sin 2\omega_2 t\left[\alpha_{22} - 2W_2\beta_{22} + \rho W_2\beta_{12}{}^2 + \frac{\rho}{2} W_2(\alpha_{12}{}^2 + \beta_{12})\right] = 0 \tag{6.16}$$

$$\cos 2\omega_2 t[-2W_1\alpha_{22} - \beta_{22} - \rho W_2\alpha_{12}\beta_{12}] = 0 \tag{6.17}$$

$$\sin(\omega_1+\omega_2)t\left[\alpha_{31} - (W_1+W_2)\beta_{31} + \frac{\rho}{2}(W_1+W_2)(\alpha_{11}\alpha_{12} - \beta_{11}\beta_{12})\right] = 0 \tag{6.18}$$

$$\cos(\omega_1+\omega_2)t\left[-(W_1+W_2)\alpha_{31} - \beta_{31} - \frac{\rho}{2}(W_1+W_2)(\alpha_{11}\beta_{12} + \alpha_{12}\beta_{11})\right] = 0\,. \tag{6.19}$$

$$\sin(\omega_1-\omega_2)t\left[\alpha_{32}+(W_1-W_2)\beta_{32}\right.$$
$$\left.+\frac{\rho}{2}(W_1-W_2)(\alpha_{11}\alpha_{12}+\beta_{11}\beta_{12})\right]=0 \qquad (6.20)$$

$$\cos(\omega_1-\omega_2)t\left[-(W_1-W_2)\alpha_{32}-\beta_{32}\right.$$
$$\left.-\frac{\rho}{2}(W_1-W_2)(\alpha_{11}\beta_{12}-\alpha_{12}\beta_{11})\right]=0\,. \qquad (6.21)$$

The solution to the systems (6.10) to (6.21) are:

$$\alpha_{11}=\frac{W_1F_1}{1+W_1^2} \qquad (6.22)$$

$$\alpha_{12}=\frac{W_1F_1}{1+W_2^2} \qquad (6.23)$$

$$\alpha_{21}=-\tfrac{1}{2}\frac{\rho F_1^2W_1}{(1+W^2)^2}\frac{4W_1^2-1}{4W_1^2+1} \qquad (6.24)$$

$$\alpha_{22}=-\tfrac{1}{2}\frac{\rho F_2^2W_2}{(1+W_2^2)^2}\frac{4W_2^2-1}{4W_2^2+1} \qquad (6.25)$$

$$\alpha_{31}=-\tfrac{1}{2}\frac{\rho F_1F_2(W_1+W_2)}{(1+W_1^2)(1+W_2^2)}\left[\frac{W_1W_2-2}{1+(W_1+W_2)^2}+1\right] \qquad (6.26)$$

$$\alpha_{32}=\tfrac{1}{2}\frac{\rho F_1F_2(W_1-W_2)}{(1+W_1^2)(1+W_2^2)}\left[\frac{W_2W_2+2}{1+(W_1-W_2)^2}+1\right] \qquad (6.27)$$

$$\beta_{11}=\frac{F_1}{1+W_1^2} \qquad (6.28)$$

$$\beta_{12}=\frac{F_2}{1+W_2^2} \qquad (6.29)$$

$$\beta_{21}=-\frac{2\rho F_1^2W_1^2}{(1+4W_1^2)(1+W_1^2)^2} \qquad (6.30)$$

$$\beta_{22}=\frac{-2\rho F_2^2W_2^2}{(1+4W_2)^2(1+W_2^2)^2} \qquad (6.31)$$

$$\beta_{31}=\tfrac{1}{2}\frac{\rho F_1F_2(W_1+W_2)^2(W_1W_2-2)}{[1+(W_1+W_2)^2](1+W_1^2)(1+W_2^2)} \qquad (6.32)$$

$$\beta_{32}=-\tfrac{1}{2}\frac{\rho F_1F_2(W_1-W_2)^2(W_1W_2+2)}{[1+(W_1-W_2)^2](1+W_1^2)(1+W_2^2)}\,. \qquad (6.33)$$

(i) *Amplification*

From (6.22) to (6.33) we may deduce some information concerning amplification.

The power of the output in the frequency of ω_1 or ω_2 is respectively

$$P_{\omega_1}=\alpha_{11}^2+\beta_{11}^2+\alpha_{21}^2+\beta_{21}^2 \qquad (6.34)$$

$$P_{\omega_2}=\alpha_{12}^2+\beta_{12}^2+\alpha_{22}^2+\beta_{22}^2 \qquad (6.35)$$

correct to within terms of order ρ^4. Then

$$P_{\omega_1}=\frac{F_1^2}{1+W_1^2}+\frac{\rho^2F_1^4W_1^2}{4(1+W_1^2)^4}\frac{(4W_1^2-1)^2+(4W_1)^2}{(4W_1^2+1)^2} \qquad (6.36)$$

and

$$P_{\omega_2}=\frac{F_2^2}{1+W_1^2}+\frac{\rho^2F_2^4W_2^2}{4(1+W_2^2)^4}\frac{(4W_2^2-1)^2+(4W_2)^2}{(4W_2^2+1)^2}\,. \qquad (6.37)$$

Thus there is no amplification to within terms of order ρ^4. (One can easily see that there will be higher order amplification.)

(ii) *Conversion*

In a similar manner, we have

$$P_{\omega_1+\omega_2}=\frac{\rho^2F_1^2F_2^2(W_1+W_2)^2}{4(1+W_1^2)^2(1+W_2^2)^2}$$
$$\left[\left(\frac{W_1W_2-2}{1+(W_1+W_2)^2}+1\right)^2+\frac{(W_1+W_2)^2(W_1W_2-2)^2}{(1+(W_1+W_2)^2)^2}\right] \qquad (6.38)$$

$$P_{\omega_1-\omega_2}=\frac{\rho^2F_1^2F_2^2(W_1-W_2)^2}{4(1+W_1^2)^2(1+W_2^2)^2}$$
$$\left[\left(\frac{W_1W_2+2}{1+(W_1-W_2)^2}+1\right)^2+\frac{(W_1-W_2)^2(W_1W_2+2)^2}{(1+(W_1-W_2)^2)^2}\right]. \qquad (6.39)$$

Thus there is conversion of energy in the second order of ρ.

To determine the response relations from the $\alpha_{11},\ldots,\beta_{32}$, we proceed as follows. In Section 5 we introduce as a parameter the value of v at $t=0$. Namely,

$$v(0)=A\,. \qquad (6.40)$$

(In actuality, A was the first Taylor coefficient of the first Fourier coefficient of v.) This led to the relation (4.31) for W. In our present situation we have two frequencies, W_1 and W_2. To obtain a second relation, we must introduce another parameter, viz.,

$$v'(0)=B\,. \qquad (6.41)$$

Of course in general one cannot prescribe two initial data for the solution of a first-order differential equation. However, we are not really doing this, because we are actually prescribing no data for $v(t)$, but rather only a periodicity requirement. The two conditions (6.40) and (6.41) may be viewed as conditions relating W_1 and W_2 to v.

If we insert v to first order in ρ into (6.40) and (6.41) we get the relations

$$A=\beta_{11}+\beta_{12} \qquad (6.42)$$

and

$$B=\alpha_{11}\omega_1+\alpha_{12}\omega_2\,. \qquad (6.43)$$

If we now assume that

$$W_1=W_{10}+\rho W_{11}+\ldots \qquad (6.44)$$

and

$$W_2=W_{20}+\rho W_{21}+\ldots, \qquad (6.45)$$

(6.40) and (6.41) become in terms of the W_{ij}

$$A=\frac{F_1}{1+W_{10}^2}+\frac{F_2}{1+W_{20}^2}\tag{6.46}$$

and

$$cB=\frac{W_{10}^2\,F_1}{1+W_{10}^2}+\frac{W_{20}^2\,F_2}{1+W_{20}^2}\,.\tag{6.47}$$

If we solve (6.46) for W_{10}^2 and insert the result into (6.47), we obtain the equation

$$(1+W_{20}^2)(F_1+F_2-A-cB)=0\,.\tag{6.48}$$

This determines a relation between A and B, which in turn suggests that

$$W_{20}\equiv W_2\tag{6.49}$$

is to be taken as prescribed *a priori.* Thus from (6.46)

$$1+W_{10}^2=\frac{F_1}{A-\dfrac{F_2}{1+W_2^2}}\,,\tag{6.50}$$

which is the zero-th order approximation to the response relation.

The next approximation to the response relation is obtained by setting $\left.\dfrac{dv}{d\rho}\right|_{\rho=0}$ equal to zero. This yields

$$W_{11}=\frac{(1+W_{10}^2)^2}{2F_1W_{10}}\Bigg[-\frac{4F_1^2W_{10}^2}{(1+4W_{10}^2)(1+W_{10}^2)^2}-\frac{4F_2^2W_2^2}{(1+4W_2^2)(1+W_2^2)^2}+\frac{F_1F_2}{(1+W_{10}^2)(1+W_2^2)}\left\{\frac{(W_{10}+W_2)^2(W_{10}W_2-2)}{1+(W_{10}+W_2)^2}-\frac{(W_{10}-W_2)^2(W_{10}W_2+2)}{1+(W_{10}-W_2)^2}\right\}\Bigg].\tag{6.51}$$

Inserting (6.50) and (6.51) into (6.44) would give us an expression for W_{11} in terms of F_1, F_2, W_2, and A correct to order ρ^2. Indeed (6.50) into (6.51) gives

$$W_{11}=\frac{\left[\dfrac{F_1}{A-\dfrac{F_2}{1+W_2^2}}\right]^2}{2F_1\sqrt{\dfrac{F_1}{A-\dfrac{F_2}{1+W_2^2}}-1}}\left[\frac{-4F_1^2\left(\dfrac{F_1}{A-\dfrac{F_2}{1+W_2^2}}-1\right)}{\left(-3+\dfrac{4F_1}{A-\dfrac{F_2}{1+W_2^2}}\right)\left(\dfrac{F_1}{A-\dfrac{F_2}{1+W_2^2}}\right)^2}-\frac{4F_2^2\,W_2^2}{(1-W_2^2)(1+W_2^2)^2}\right.$$
$$\left.+\frac{4F_1F_2}{\left(\dfrac{F_1}{A-\dfrac{F_2}{1+W_2^2}}\right)(1+W_2^2)}\left\{\frac{\left(\left(\dfrac{F_1}{A-\dfrac{F_2}{1+W_2^2}}-1\right)(W_2^2+W_2-1)+\left(\dfrac{F_1}{A-\dfrac{F_2}{1+W_2^2}}-1\right)(W_2^4-1)-W_2^2(1+W_2^2)\right)}{1+2\left(\dfrac{F_1}{A-\dfrac{F_2}{1+W_2^2}}\right)^2+2W_2^2+\left[\left(\dfrac{F_1}{A-\dfrac{F_2}{1+W_2^2}}\right)^2-W_2^2\right]^2}\right\}\right]\tag{6.52}$$

Combining (6.44), (6.50), and (6.52) gives a relation between W_1, W_2, F_1, F_2, ρ and A which is the response relation. This relation determines the value of the periodic solution v at $t=0$, A, in terms of the driving frequencies, their amplitudes, and the degree of nonlinearity, ρ.

IV. The difference-differential equation

7. *Periodic solutions under a commensurability restriction*

In this section we will obtain periodic solutions of

$$c(v)v_t\Big|_{t-\tau}^{t}=-v(t)+\phi(t-\tfrac{1}{2}\tau)\tag{7.1}$$

when τ and the period T of ϕ are commensurable.

(i) *Harmonics*

Let p and q be relatively prime integers, such that

$$p\tau=qT.\tag{7.2}$$

Now consider

$$c(v)v_t\Big|_{t-2\tau}^{t-\tau}=\phi(t-\tau-\tfrac{1}{2}\tau)-v(t-\tau)$$
$$\cdots$$
$$c(v)v_t\Big|_{t-p\tau}^{t-(p-1)\tau}=(t-(p-1)\tau-\tfrac{1}{2}\tau)-v(t-(p-1)\tau)\,.\tag{7.3}$$

Suppose (7.1) has a solution of period T. Then this solution satisfies each equation in (7.3). It must then satisfy the sum of all equations in (7.1) and (7.3). This sum is

$$c(v)v_t\Big|_{t-p\tau}^{t}=\sum_{j=0}^{p-1}\phi(t-j\tau-1/2\tau)-v(t-j\tau)\,. \qquad (7.4)$$

Since $p\tau=qT$, this reduces to

$$0=\sum_{j=0}^{p-1}\phi(t-j\tau-1/2\tau)-v(t-j\tau)\,. \qquad (7.5)$$

Thus the totality of solutions of (7.5) contains the periodic solutions (with period T) of (7.1). The totality of solutions of (7.5) is a linear combination of exponentials plus the particular solution $\phi(t-1/2\tau)$. Specifically,

$$v(t)=\phi(t-1/2\tau)+\sum_{n=1}^{p-1}A_ne^{-2\pi\,\mathrm{i}nt/p}\,. \qquad (7.6)$$

Not all of the terms in (7.6) have period T, nor are all of them real. Indeed not all of the functions represented by (7.6) are solutions of (7.1). To determine which of the expressions in (7.6) satisfy the three properties just mentioned requires special choices of τ and the A_n. We will consider these possibilities in general and in particular when $\phi=F\cos\omega t$, and $c(v)=c(1+\rho v)$, (here c and ρ are constants).

Case 1: Suppose that τ is a multiple of T, i.e., $p=1$. Then (7.6) reduces to

$$v(t)=\phi\left(t-\frac{1}{2}\tau\right). \qquad (7.7)$$

This is real and has period T. For it to be a solution of (7.1) requires that

$$c\left[\phi\left(t-\frac{1}{2}\tau\right)\right]\phi'\left(t-\frac{1}{2}\tau\right)\Bigg|_{t-\tau}^{t}=0\,, \qquad (7.8)$$

or in our special case where $c(v)=c(1+\rho v)$,

$$cF\omega\left[1+\rho F\cos\omega\left(t-\frac{1}{2}\tau\right)\sin\omega\left(t-\frac{1}{2}\tau\right)-\left\{1+\rho F\cos\omega\left(t-\frac{3}{2}\tau\right)\sin\omega\left(t-\frac{3}{2}\tau\right)\right\}\right]=0\,. \qquad (7.9)$$

Letting $\omega t=\theta,\ \dfrac{\omega\tau}{2}=\psi$,

(7.9) becomes

$$\sin\theta\cos\psi-\cos\theta\sin\psi-\sin\theta\cos3\psi+\cos\theta\sin3\psi+\frac{\rho F}{2}[\sin2\theta\cos2\psi-\cos2\theta\sin2\psi-\sin2\theta\cos6\psi+\cos2\theta\sin6\psi]=0\,. \qquad (7.10)$$

Then we must have

$$\cos\psi-\cos3\psi=0\,,$$
$$\sin\psi-\sin3\psi=0\,,$$
$$\cos2\psi-\cos6\psi=0\,,$$
$$\sin2\psi-\sin6\psi=0\,. \qquad (7.11)$$

The equations (7.11) each imply

$$\psi=n\pi\,, \qquad (7.12)$$

i.e.,

$$\omega\tau=2n\pi \qquad (7.13)$$

or $\quad n=q$.

$\omega\tau=2q\pi$ is a condition on both ω and τ in order that (7.1) have a periodic solution.

Case 2: If p is larger than one, so that the sum in (7.6) is not empty, then in order to make (7.6) real we must have certain relations among the A_n.

If p is even, we must have

$$v(t)=\phi\left(t-\frac{1}{2}\tau\right)+\sum_{n=1}^{\frac{p}{2}-1}2A_n\cos\frac{2\pi nt}{p\tau}\,. \qquad (7.14)$$

If p is odd, we have

$$v(t)=\phi\left(t-\frac{1}{2}\tau\right)+\sum_{n=1}^{\frac{p-1}{2}}2A_n\cos\frac{2\pi nt}{p\tau}\,. \qquad (7.15)$$

The terms in the sums here will be periodic of period T, if

$$\frac{nT}{p\tau}=\sigma=\text{integer}\,, \qquad (7.16)$$

i.e., if

$$n=q\sigma\,, \qquad (7.17)$$

i.e., if q/n (q divides n).

Thus we must first of all have, since $n>p$, that $q<p$. If this is not the case, as in Case 1 above where $p=1$, we will have $\phi(t-1/2\tau)$ as the only possible periodic solution of (7.1). We may thus rewrite (7.14) and (7.15) as

$$v(t)=\phi\left(t-\frac{1}{2}\tau\right)+2\sum_{\substack{n=\sigma q\\ \sigma>0}}^{\frac{p}{2}-1,\ \frac{p-1}{2}}A_n\cos\frac{2\pi nt}{p\tau}\,. \qquad (7.18)$$

The upper limit, which is an integer, is taken in this expression.

Now we must insert this into (7.1). This yields

$$c\left[\phi\left(t-\frac{1}{2}\tau\right)+2\sum_{\substack{n=\sigma q\\ \sigma>0}}^{\frac{p}{2}-1,\ \frac{p-1}{2}}A_n\cos\frac{2\pi nt}{p\tau}\right]\left[\phi'\left(t-\frac{1}{2}\tau\right)-\frac{4\pi}{p}\sum_{\substack{n=\sigma q\\ \sigma>0}}^{\frac{p}{2}-1,\ \frac{p-1}{2}}nA_n\sin\frac{2\pi nt}{p\tau}\right]\Bigg|_{t-\tau}^{t}\,.$$

$$=-2\sum_{\substack{n=\sigma q\\ \sigma>0}}^{\frac{p}{2}-1,\ \frac{p-1}{2}}A_n\cos\frac{2\pi nt}{p\tau}\,. \qquad (7.19)$$

Let us specialize c and ϕ as before. We get

$$c\left[1+F\cos\omega\left(t-\frac{1}{2}\tau\right)+2\rho\sum_{\substack{n=\sigma q\\ \sigma>0}}^{\frac{p}{2}-1,\ \frac{p-1}{2}} A_n\cos\frac{2\pi nt}{p\tau}\right]\left[-\omega F\sin\omega\left(t-\frac{1}{2}\tau\right)-\frac{4\pi}{p\tau}\sum nA_n\sin\frac{2\pi nt}{p\tau}\right]$$

$$-c\left[1+\rho F\cos\omega\left(t-\frac{3}{2}\tau\right)+2\rho\sum A_n\cos\frac{2\pi n(t-\tau)}{p\tau}\right]\left[-\omega F\sin\omega\left(t-\frac{3}{2}\tau\right)-\frac{4\pi}{p\tau}\sum nA_n\sin\frac{2\pi n(t-\tau)}{p\tau}\right]$$

$$=-2\sum A_n\cos\frac{2\pi nt}{p\tau}\,. \tag{7.20}$$

To explore the possibilities, let us suppose that the sum reduces to a single entry. Setting the coefficients of $\sin\omega t$ and $\cos\omega t$ in the resulting expression equal to zero gives respectively,

$$\sin\omega t\left[c\omega F\left(-\cos\frac{1}{2}\omega\tau+\cos\frac{3}{2}\omega\tau\right)\right]=0 \tag{7.21}$$

and

$$\cos\omega t\left[c\omega F\left(\sin\frac{1}{2}\omega\tau-\sin\frac{3}{2}\omega\tau\right)\right]=0\,. \tag{7.22}$$

The solution to these equations is

$$\frac{1}{2}\omega\tau=n\pi \tag{7.23}$$

or

$$\frac{q}{p}=n\,. \tag{7.24}$$

This is impossible since $p>q$. Thus we must have more terms which are sines and cosines of ωt. The only way of producing such terms is to set

$$\frac{2\pi n}{p\tau}=\omega\,. \tag{7.25}$$

This implies that

$$n=q\,, \tag{7.26}$$

and

$$\omega\tau=2\pi\frac{q}{p}\,. \tag{7.27}$$

Thus q must be less than or equal to $p/2-1$ or $p/2-1/2$, whichever is an integer.

With this restriction on $\omega\tau$ and n, equating coefficients of sines and cosines to zero as before, gives

$$-\cos\frac{1}{2}\omega\tau+\cos\frac{3}{2}\omega\tau-\frac{4\pi nA_n}{\omega Fp\tau}=0 \tag{7.28}$$

and

$$\sin\frac{1}{2}\omega\tau-\sin\frac{3}{2}\omega\tau-\frac{4\pi nA_n}{p\tau\omega F}\sin\frac{2\pi n}{p}+\frac{2A_n}{c\omega F}=0\,. \tag{7.29}$$

Inserting (7.26) and (7.27) into (7.28) and (7.29) gives

$$-\cos\frac{\pi q}{p}+\cos\frac{3\pi q}{p}-\frac{2A_q}{F}=0\,, \tag{7.30}$$

and

$$\sin\frac{\pi q}{p}-\sin 3\pi\frac{q}{p}-2\frac{A_q}{F}\sin 2\pi\frac{q}{p}+\frac{2A_q}{c\omega F}=0\,. \tag{7.31}$$

These equations combine to yield

$$A_q=\frac{F}{2}\left(-\cos\pi\frac{q}{p}+\cos 3\pi\frac{q}{p}\right)=\frac{F(1+c\omega)}{2(1+\sin 2\pi q/p)}\left(\sin\frac{\pi q}{p}-\sin 3\frac{\pi q}{p}\right). \tag{7.32}$$

(7.27) and (7.32) are the response relations for ω, τ and A_q. We may conclude from this example that a periodic solution of (7.1) exists only when certain relations are satisfied by ω and τ. This example here will serve to guide our perturbational analysis of (7.1), which we will explore in Section 4.

We now inquire into the questions of subharmonics.

(ii) *Subharmonics*

The analysis of this question proceeds in a manner similar to (i). Thus we merely make the following observations:

If we want a solution $v(t)$ of (7.1), with period rT, but with no smaller period, then the number p in equation (7.5) should be replaced by a multiple of itself if $r\nmid q$. Indeed p is replaced by rp. Then (7.6) becomes

$$v(t)=\phi\left(t-\frac{1}{2}\tau\right)+\sum_{n=1}^{rp-1}A_n\exp\left(-\frac{2\pi\,in}{rp\tau}t\right). \tag{7.33}$$

The existence of subharmonics reduces to the question of whether the sum in (7.33) has terms with period rT which are not periodic with a smaller period of the form sT. This will be the case if

$$\frac{nrT}{rp\tau}=\text{integer} \tag{7.34}$$

but

$$\frac{nsT}{rp\tau}\neq\text{integer}\,. \tag{7.35}$$

These conditions reduce to

$$\frac{n}{pq} = \text{integer}\,, \tag{7.36}$$

and

$$\frac{n}{pq}\frac{s}{r} \neq \text{integer}\,, \tag{7.37}$$

respectively. Thus we must have

$$pq|n\,, \tag{7.38}$$

but

$$pqr\nmid n\,. \tag{7.39}$$

This is the case if and only if $r\nmid n$, since $r\nmid s$.

• 8. *Perturbational analysis*

In this section we will derive the response relation for periodic solutions of our equation. We do this without imposing a commensurability restriction on T and τ.

For definiteness we take

$$\phi(t)=F\cos\omega t\,, \tag{8.1}$$

and

$$c(v)=c(1+\rho v)\,, \tag{8.2}$$

where ρ and c are constants and ρ is considered to be small.

Now we introduce the transformation

$$\theta=\omega t\,,\quad \psi=\omega\tau\,,\quad W=\omega c\,,\quad u(\theta)=v(t)\,. \tag{8.3}$$

Our equation then becomes

$$W(1+\rho u(\theta))u_\theta(\theta)-W(1+\rho u(\theta-\psi))u_\theta(\theta-\psi)$$

$$=-u(\theta)-F\cos(\theta-\tfrac{1}{2}\psi)\,. \tag{8.4}$$

We introduce two auxiliary parameters in

$$u(0)=A\,, \tag{8.5}$$

and

$$u'(0)=B\,. \tag{8.6}$$

(i) *Harmonic response*

From what we have deduced in Section 2, we know that a periodic solution of (8.4) will exist only if certain relations are satisfied by ω and τ (or W and ψ). From our experience in Section 2, we may expect these relations to take the form of W and ψ as functions of the initial conditions, A and B. Thus in performing perturbational analysis we are led to look for u, W, and ψ in the form

$$u=u_0+\rho u_1+\ldots \tag{8.7}$$

$$W=W_0+\rho W_1+\ldots \tag{8.8}$$

$$\psi=\psi_0+\rho\psi_1+\ldots\quad. \tag{8.9}$$

In addition we assign A and B among the coefficients in (8.7) through

$$u_0(0)=A\,,\quad u_i(0)=0\,,\quad i>0\,,$$

$$u_0'(0)=B\,,\quad u_i'(0)=0\,,\quad i>0\,. \tag{8.10}$$

The first two perturbation equations are respectively

$$W_0[u_0'(\theta)-u_0'(\theta-\psi_0)]=-u_0(\theta)-F\cos(\theta-1/2\psi_0) \tag{8.11}$$

$$W_0[u_1'(\theta)-u_1'(\theta-\psi_0)]=-u_1(\theta)-u_0'(W_1+W_0u_0)\Big|_{\theta-\psi_0}^{\theta}$$

$$-W_0\psi_1u_0''(\theta-\psi_0)-F\frac{\psi_1}{2}\sin(\theta-1/2\psi_0)\,. \tag{8.12}$$

We look for a solution of (8.11) in the form

$$u_0=\alpha_0\sin\theta+\beta_0\cos\theta\,. \tag{8.13}$$

From (8.10) we see that $\alpha_0=B$ and $\beta_0=A$. Thus if we insert (8.13) into (8.11) and equate coefficients of $\sin\theta$ and $\cos\theta$ to zero, we obtain two equations involving the four quantities α_0, β_0, W_0 and ψ_0. These equations are

$$W_0[-\beta_0-\alpha_0\sin\psi_0+\beta_0\cos\psi_0]+\alpha_0+F\sin\tfrac{1}{2}\psi_0=0 \tag{8.14}$$

and

$$W_0[\alpha_0-\alpha_0\cos\psi_0-\beta_0\sin\psi_0]+\beta_0+F\cos\tfrac{1}{2}\psi_0=0\,. \tag{8.15}$$

Since α_0 and β_0 are known in terms of A and B, we view (8.14) and (8.15) as a pair of simultaneous transcendental equations for the determination of W_0 and ψ_0. If W_0 is eliminated from the pair (8.14) and (8.15), there results, after some manipulation, a quadratic in $\sin\psi_0/2$. The quadratic has an extraneous null root. The remaining root is

$$\sin\frac{\psi_0}{2}=\frac{-\alpha_0F}{(\alpha_0^2+\beta_0^2)}=\frac{-BF}{(A^2+B^2)}\,. \tag{8.16}$$

With this information, W_0 is readily found to be

$$W_0=\frac{(A^2+B^2)(B^2+A^2-F^2)}{-2BF[AF\pm\sqrt{(A^2+B^2)^2-B^2F}]}\,. \tag{8.17}$$

Continuing our procedure by inserting (8.13) into (8.12), leads us to look for a solution of (8.12) in the form

$$u_1=\alpha_{11}\sin\theta+\beta_{11}\cos\theta+\alpha_{12}\sin2\theta+\beta_{12}\cos2\theta\,. \tag{8.18}$$

The equations for α_{11}, β_{11}, α_{12}, and β_{12} are linear.

After solving these equations, the initial conditions (8.10) give the following linear equations for determining ψ_1 and W_1.

$$0=\beta_{11}+\beta_{12}$$

$$0=\alpha_{11}+2\alpha_{12}\,. \tag{8.19}$$

Once ψ_1 and W_1 are determined, we can combine all of our results and obtain

$$W=W_0(A,B)+\rho W_1(A,B) \tag{8.20}$$

and

$$\psi=\psi_0(A,B)+\rho\psi_1(A,B)\,, \tag{8.21}$$

which are the response relations. Thus for each value of driving frequency, W, and each value of ψ, one or more pairs (A,B) which fix the magnitude and phase of the harmonic u are determined.

A summary of these manipulations for obtaining ψ_1 and W_1 is:

$$W_1=(Z_3Y_4-Z_4Y_3)/(X_4Y_3)\,, \tag{8.22}$$

$$\psi_1=(Z_4X_3-X_4Z_3)/(X_3Y_4-X_4Y_3)\,, \tag{8.23}$$

$$X_3=A(1-\cos\psi_0)+B\sin\psi_0\,,$$

$$Y_3=W_0B\cos\psi_0+W_0A\sin\psi_0-\frac{F}{2}\cos\frac{\psi_0}{2}\,,$$

$$X_4=-B(1-\cos\psi_0)+A\sin\psi_0\,,$$

$$Y_4=-BW_0\sin\psi_0+W_0A\cos\psi_0+\frac{F}{2}\sin\frac{\psi_0}{2}\,,$$

$$Z_3=\alpha_{11}(1-W_0\sin\psi_0)+\beta_{11}W_0(-1+\cos\psi_0)\,,$$

$$Z_4=\alpha_{11}W_0(1-\cos\psi_0)+\beta_{11}(1-W_0\sin\psi_0)\,,$$

$$-\tfrac{1}{2}\alpha_{11}=\alpha_{12}=(Z_1Y_2-Z_2Y_1)/(X_1Y_2-X_2Y_1)\,,$$

$$-\beta_{11}=\beta_{12}=(X_1Z_2-X_2Z_1)/(X_1Y_2-X_2Y_1)\,,$$

$$X_1=1-2W_0\sin 2\psi_0\,,$$

$$Y_1=2(-1+\cos 2\psi_0)\,,$$

$$X_2=2W_0(1-\cos 2\psi_0)\,,$$

$$Y_2=1-2W_0\sin 2\psi_0\,,$$

$$Z_1=\frac{W_0}{2}(A^2-B^2)(1-\cos 2\psi_0)+W_0AB\sin 2\psi_0\,,$$

$$Z_2=\frac{W_0}{2}(A^2-B^2)\sin 2\psi_0+W_0AB\cos 2\psi_0-ABW_0\,.$$

In Figs. 9 and 10 we plot the response curves.

(ii) *Subharmonic response*

Let us obtain the first harmonic. Thus in (8.11) insert

$$u_0=\alpha_{01}\sin\frac{\theta}{2}+\beta_{01}\cos\frac{\theta}{2}+\alpha_{02}\sin\theta+\beta_{02}\cos\theta\,. \tag{8.24}$$

We obtain

$$\begin{aligned}W_0\Big[&\frac{\alpha_{01}}{2}\cos\frac{\theta}{2}-\frac{\beta_{01}}{2}\sin\frac{\theta}{2}+\alpha_{02}\cos\theta-\beta_{02}\sin\theta\\&-\frac{\alpha_{01}}{2}\cos\frac{\theta-\psi_0}{2}+\frac{\beta_{01}}{2}\sin\frac{\theta-\psi_0}{2}\\&-\alpha_{02}\cos(\theta-\psi_0)+\beta_{02}\sin(\theta-\psi_0)\Big]\\&=-\alpha_{01}\sin\frac{\theta}{2}-\beta_{01}\cos\frac{\theta}{2}-\alpha_{02}\sin\theta\\&\quad-\beta_{02}\cos\theta+F\cos(\theta-\tfrac{1}{2}\psi_0)\,.\end{aligned} \tag{8.25}$$

Then equating coefficients of $\sin\theta/2$, $\cos\theta/2$, $\sin\theta$, and $\cos\theta$ to zero gives

$$\sin\frac{\theta}{2}\Big[-W_0\frac{\beta_{01}}{2}-W_0\frac{\alpha_{01}}{2}\sin\frac{\psi_0}{2}+W_0\frac{\beta_{01}}{2}\cos\frac{\psi_0}{2}+\alpha_{01}\Big]=0\,, \tag{8.26}$$

$$\cos\frac{\theta}{2}\Big[W_0\frac{\alpha_{01}}{2}-W_0\frac{\alpha_{01}}{2}\cos\frac{\psi_0}{2}-W_0\frac{\beta_{01}}{2}\sin\frac{\psi_0}{2}+\beta_{01}\Big]=0\,, \tag{8.27}$$

$$\sin\theta[-W_0\beta_{02}-W_0\alpha_{02}\sin\psi_0+W_0\beta_{02}\cos\psi_0+\alpha_{02}-F\sin\tfrac{1}{2}\psi_0]=0\,, \tag{8.28}$$

and

$$\cos\theta[W_0\alpha_{02}-W_0\alpha_{02}\cos\psi_0-W_0\beta_{02}\sin\psi_0+\beta_{02}-F\cos\tfrac{1}{2}\psi_0]=0\,. \tag{8.29}$$

These equations together with

$$\beta_{01}+\beta_{02}=A\,, \tag{8.30}$$

and

$$\frac{\alpha_{01}}{2}+\alpha_{02}=B\,, \tag{8.31}$$

which are obtained from (8.5), (8.6) and (8.24) form a system of six equations for the six unknowns, α_{01}, α_{02}, β_{01}, β_{02}, W_0 and ψ_0.

A procedure for solving these equations is given as follows. First we note that (8.26) and (8.27) are linear homogeneous in α_{01} and β_{01} with coefficients given in terms of W_0 and ψ_0. In order to find a nontrivial solution of (8.26) and (8.27) the determinant of the coefficient matrix of this system must vanish. This gives a relation between W_0 and ψ_0, viz.,

$$\left(1-\frac{W_0}{2}\sin\frac{\psi_0}{2}\right)^2-\frac{W_0^2}{4}\left(1-\cos\frac{\psi_0}{2}\right)^2=0\,. \tag{8.32}$$

Now equation (8.27), say, may be solved for β_{01} in terms of α_{01}, ψ_0 and W_0 and provided

$$(1-W_0\sin\psi_0)^2-W_0^2(1-\cos\psi_0)^2\neq 0\,, \tag{8.33}$$

(8.28) and (8.29) may be solved for α_{02} and β_{02} in terms of W_0 and ψ_0. When these values of β_{01}, α_{02}, and β_{02} are inserted into (8.30) and (8.31), we have two equations for α_{01}, W_0 and ψ_0. Eliminating α_{01} gives

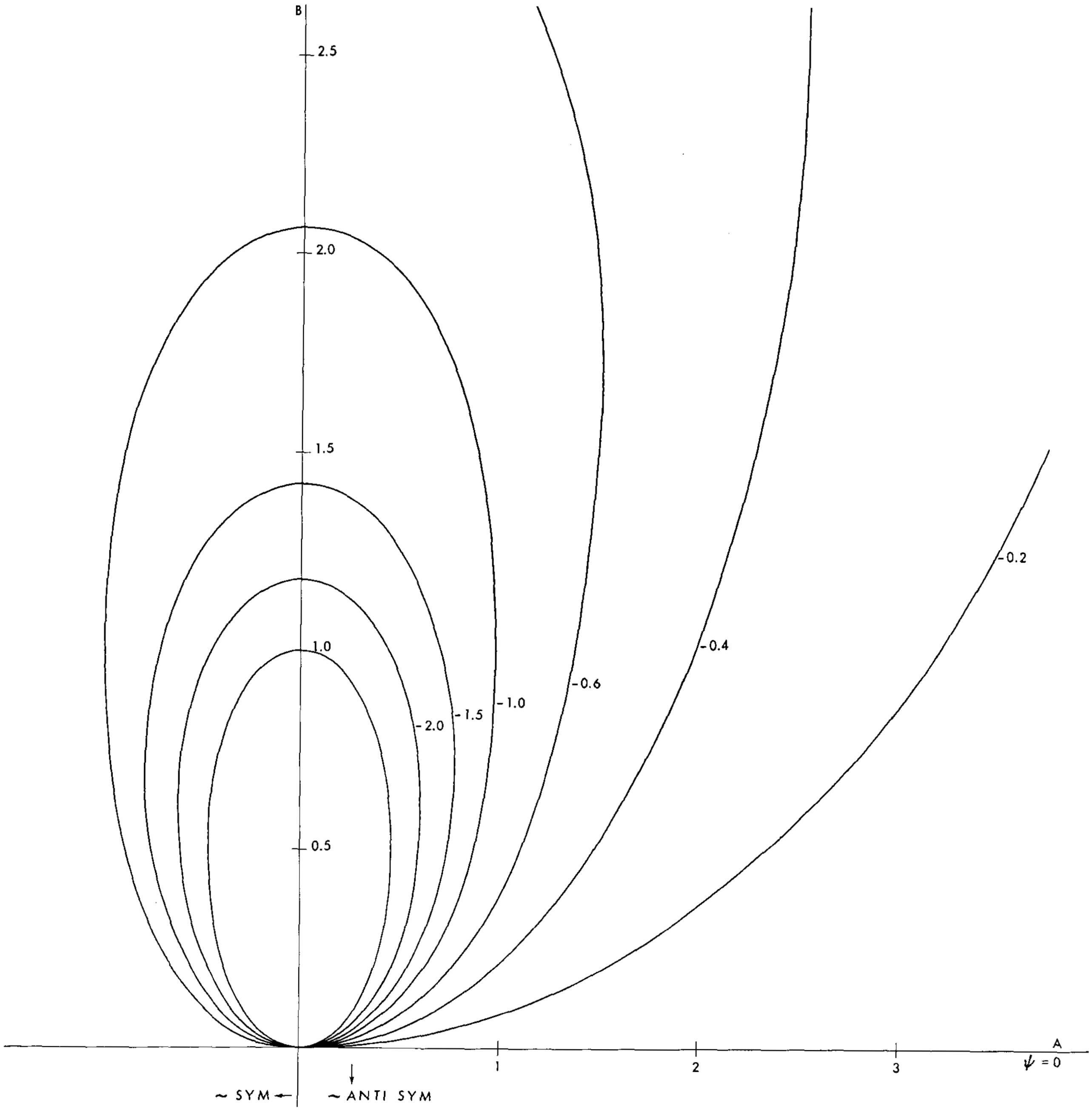
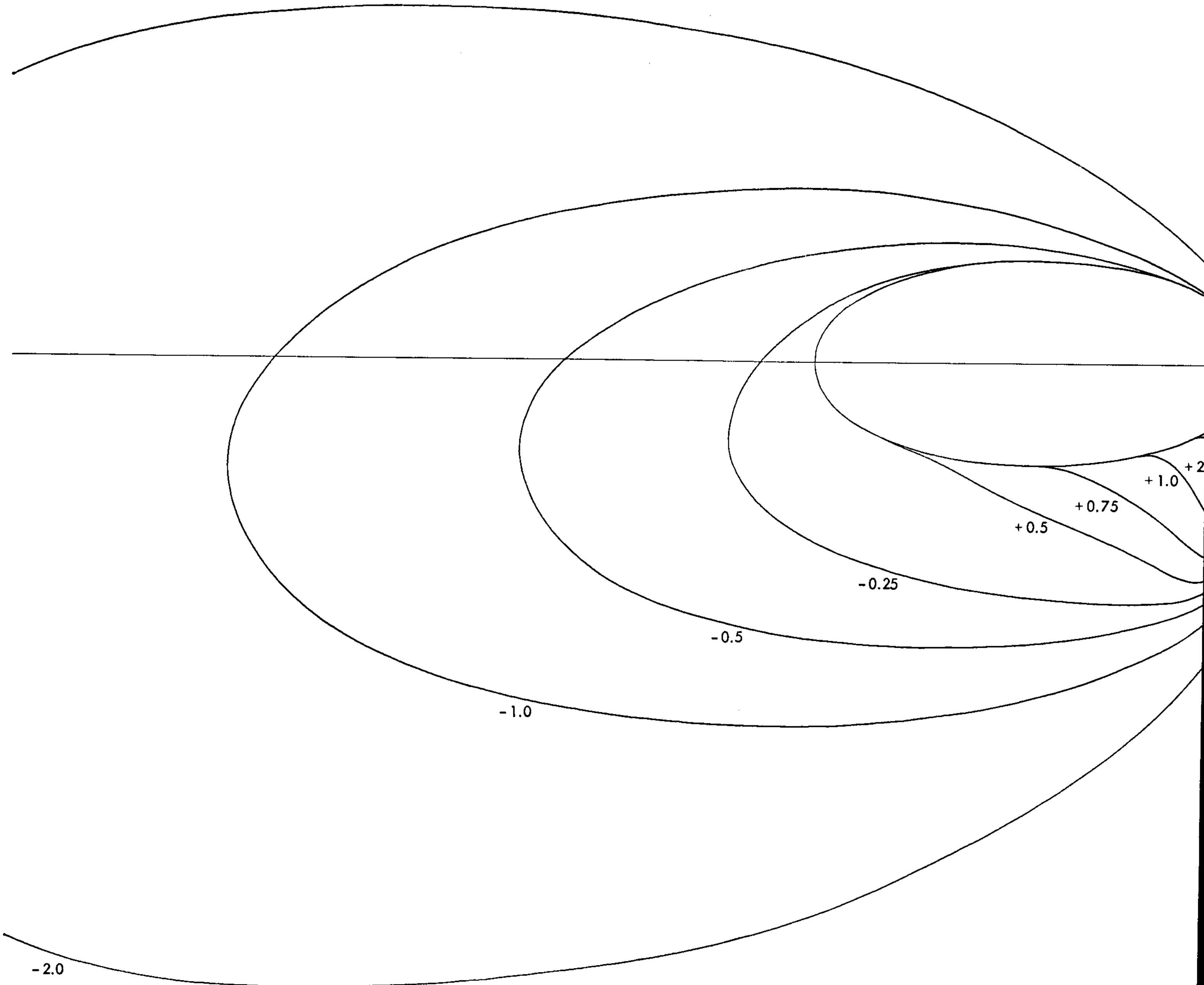


Figure 9

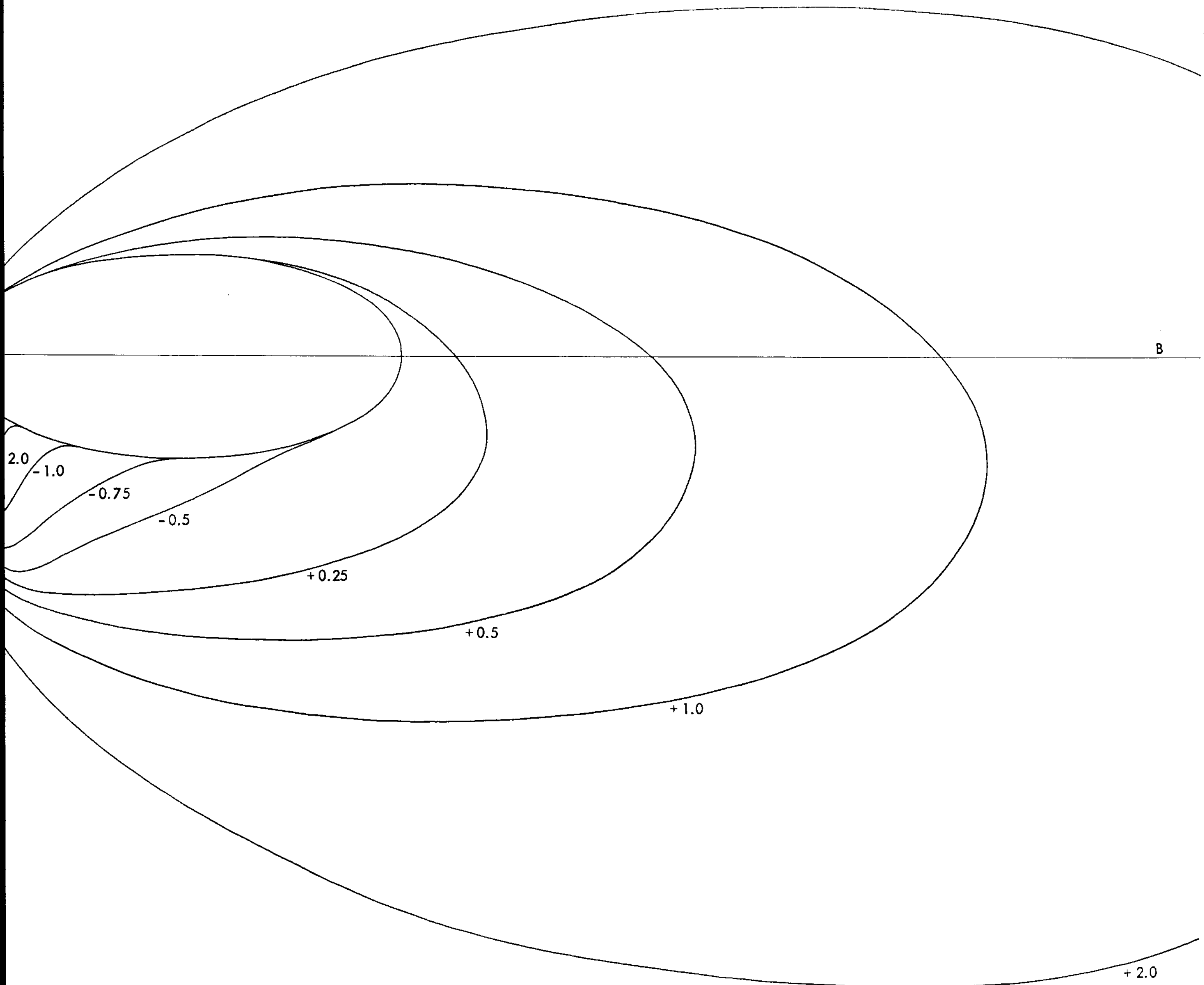
$$-W_0\frac{1-\cos\psi_0}{1-\dfrac{W_0}{2}\sin\dfrac{\psi_0}{2}}=\frac{A-F\dfrac{[-\cos\frac{1}{2}\psi_0(1-W_0\sin\psi_0)-W_0\sin\frac{1}{2}\psi_0(1-\cos\psi_0)]}{(1-W_0\sin\psi_0)^2-W_0^2(1-\cos\psi_0)^2}}{B-F\dfrac{[\sin\frac{1}{2}\psi_0(1-W_0\sin\psi_0)-W_0\cos\frac{1}{2}\psi_0(1-W_0\sin\psi_0)]}{(1-W_0\sin\psi_0)^2-W_0^2(1-\cos\psi_0)^2}}. \tag{8.34}$$

Equations (8.32) and (8.34) must now be solved for W_0 and ψ_0. When this is done we may compute α_{01}, β_{01}, α_{02}, and β_{02} from (8.27) to (8.29). When these operations are done, we insert u_0 in (8.24) into (8.12) and obtain an equation for u_1. We look for u_1 in the form of sines and cosines of $\theta/2$, θ, $3\theta/2$ and 2θ. The resulting equations for determining the coefficients of u_1 and ψ_1 and W_1 are all linear now. The remainder of the procedure is straightforward. We eliminate all details.

(For references and footnotes, see page 24.)



22 *Figure 10*


B
2.0
-1.0
-0.75
-0.5
+0.25
+0.5
+1.0
+2.0

References and footnotes

1. J. W. S. Rayleigh, *Theory of Sound,* vol. 1, Dover, New York; pp. 208-212.
2. J. B. Keller, "Bowing of Violin Strings," *Communications on Pure and Applied Mathematics* **6,** 483-495 (1953).

3. For certain diodes, a plot of the charge on the diode versus the voltage across it yields a curve whose slope dq/dv is a nonconstant function of voltage. This function, which is the capacitance of the diode, characterizes some of the properties of the diode as a circuit element.
4. Here and in the remainder of this paper, lettered subscripts will be used to denote differentiation.
5. The actual form of this boundary condition is not essential, since we will only treat the cases in which the load impedance is matched to the line, i.e., $l_2 = \infty$.
6. *Brouwer's Fixed Point Theorem:* A continuous map of a closed, simply connected and bounded set in Euclidean space into itself has a fixed point.
7. We remark that this flow will be area reducing for the system of differential equations (3.7) even when in the right member $-v$ is replaced by $-f(v)$, where f is any monotone increasing differentiable function of voltage.
8. The author is indebted to R. E. Gomory for this observation.
9. This is the case since the voltage variation around such a circuit is governed by the differential equation (3.1).

10. See J. J. Stoker, *Nonlinear Vibrations,* Interscience, 1950.

11. The author is indebted to R. L. Garwin for this observation.

The author is grateful to Mrs. Lenore Selfridge for performing the 704 computations which resulted in Figures 9 and 10.

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